

The 5 Color Theorem

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Euler's Formula



We start with Euler's formula for polyhedra (and, hence for connected planar graphs ...):

$$v - e + f = 2$$

where v is the number of vertices, e is the number of edges, and f is the number of faces.

We can show that this formula is correct by induction. First, though, we will change our view of a 3-dimensional polyhedron to a flat object in the plane. We poke a hole inside one face, and stretch that point / face out to infinity in all directions. What we then have is a collection of vertices, some of which are connected by edges. A face is a region surrounded by edges, with no vertices or edges in the interior. Note that there is one large exterior face surrounding everything and going off to infinity in all directions.

(do induction . . . exercise . . .)

There is a nice corollary to the Euler formula. Note that each face is bounded by at least 3 edges, and each edge is part of the boundary of a face twice (once on each side of the edge), and so

$$3f \leq 2e.$$

Now, from Euler's formula, we have

$$f = 2 - v + e$$

and so

$$3(2 - v + e) \leq 2e$$

or

$$2 - v + e \leq \frac{2}{3}e.$$

Moving things around, we have

$$\frac{1}{3}e \leq v - 2$$

or

$$e \leq 3v - 6.$$

From this, we can get a nice fact: every planar graph has a vertex of degree 5 or fewer.

We can show this by contradiction. First, note that every edge has a vertex at each end, and so the sum of all the degrees of the vertices is equal to twice the number of edges. Now, if all the degrees are greater than or equal to 6, we will have that

$$6v \leq \text{sum of degrees} = 2e$$

or

$$3v \leq e.$$

Now, putting together the two inequalities, we would have

$$3v \leq e \leq 3v - 6.$$

This is obviously a contradiction.

Therefore, it must be the case that there is at least one vertex with degree 5 or fewer.

The 6-Color Theorem



Now it is easy to prove the 6-Color Theorem.

First, let's clarify what we mean by that. A (vertex) *coloring* of a graph is an assignment of colors to the vertices of the graph in such a way that no edge has the same color assigned to the vertices at both its ends. For any n , we say that a graph can be n -colored if we can *color* the graph with n (or fewer) colors. The 6-color theorem is then:

Theorem 1. *Every planar graph can be 6-colored.*

Proof: This is now easy. We do it by induction. First, without loss of generality (WLOG), we can assume that the graph is connected. If it is not connected, we just color each connected

component. They never conflict with each other, so we will be done.

We will do the induction on the number of vertices.

Base case: The simplest connected planar graph consists of a single vertex. Pick a color for that vertex. we are done.

Induction step: Assume $k \geq 1$, and assume that every planar graph with k or fewer vertices can be 6-colored. Now consider a planar graph with $k + 1$ vertices. From above, we know that the graph has a vertex of degree 5 or fewer. Remove that vertex (and all edges connected to it). By the induction hypothesis, we can 6-color the remaining graph. Put the vertex (and edges) back in. We have a graph with every vertex colored (without conflicts) except for the "special" one. There

are at most 5 colors adjacent, so we have at least one color left. Use an available color for that vertex. We have then 6-colored the graph.

By induction, the theorem is proved. Q.E.D.

The 5-Color Theorem



Somewhat more difficult, but still not too hard, is the next theorem:

Theorem 2. *Every planar graph can be 5-colored.*

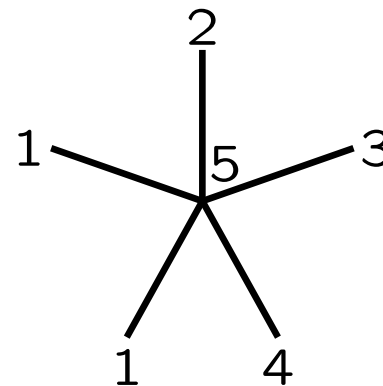
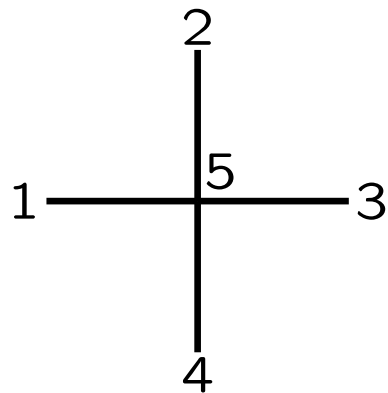
Proof: As you might expect, we will again do this by induction on the number of vertices.

Base case: The simplest connected planar graph consists of a single vertex. Pick a color for that vertex. we are done.

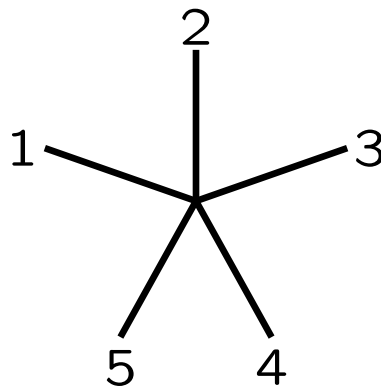
Induction step: Assume $k \geq 1$, and assume that every planar graph with k or fewer vertices can be 5-colored. Now consider a planar graph with $k + 1$ vertices. From above, we know that

the graph has a vertex of degree 5 or fewer. Remove that vertex (and all edges connected to it). By the induction hypothesis, we can 5-color the remaining graph. Put the vertex (and edges) back in. We have a graph with every vertex colored (without conflicts) except for the one.

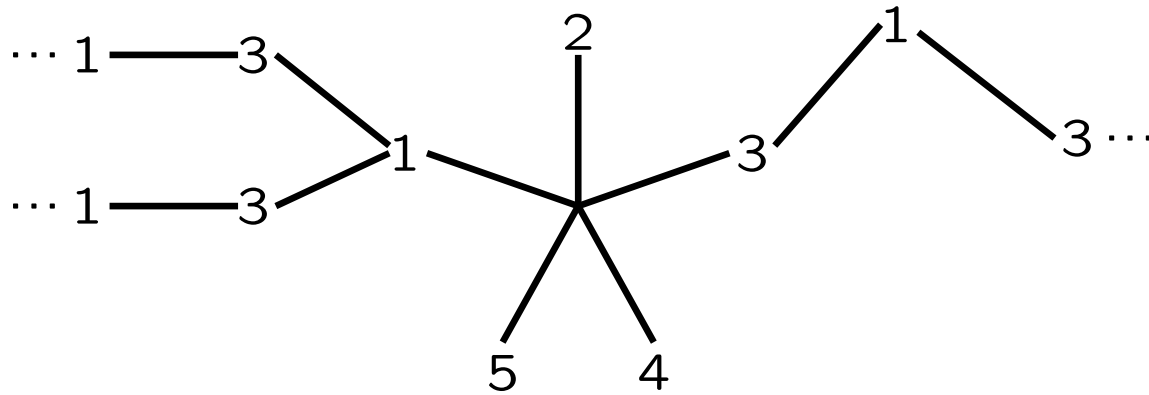
If the vertex has degree less than 5, or if it has degree 5 and only 4 or fewer colors are used for vertices connected to it, we can pick an available color for it, and we are done (numbers represent colors).



If the vertex has degree 5, and all 5 colors are connected to it, we have a little more work to do. In this case, using numbers 1 through 5 to represent colors, we label the vertices adjacent to the "special" (degree 5) vertex 1 through 5 (in order).

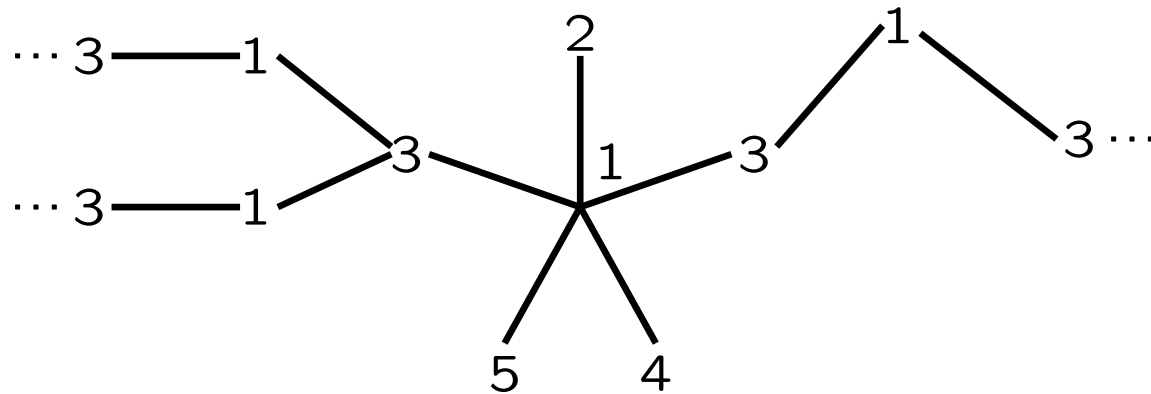


Now make a subgraph out of all the vertices colored 1 or 3 which are connected to the 1 and 3 colored vertices adjacent to the "special" vertex.

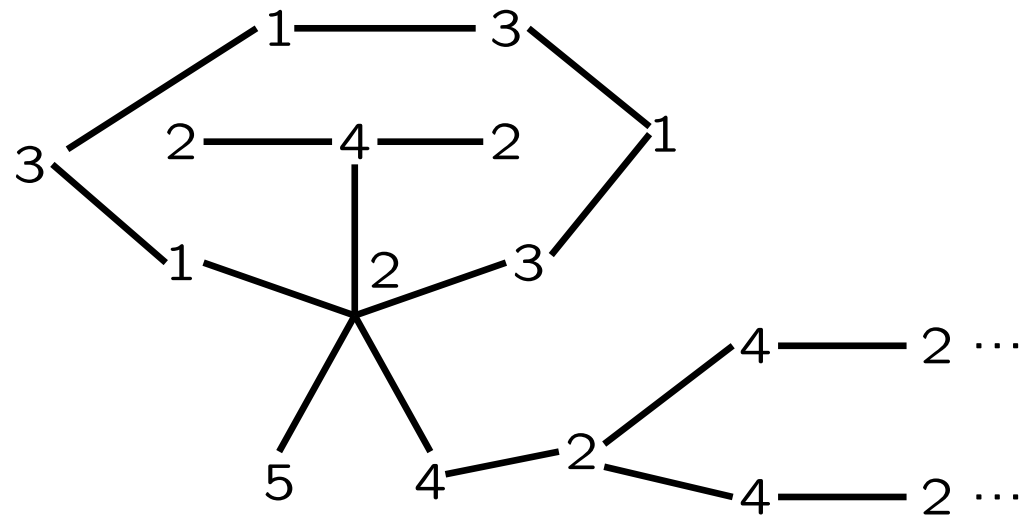
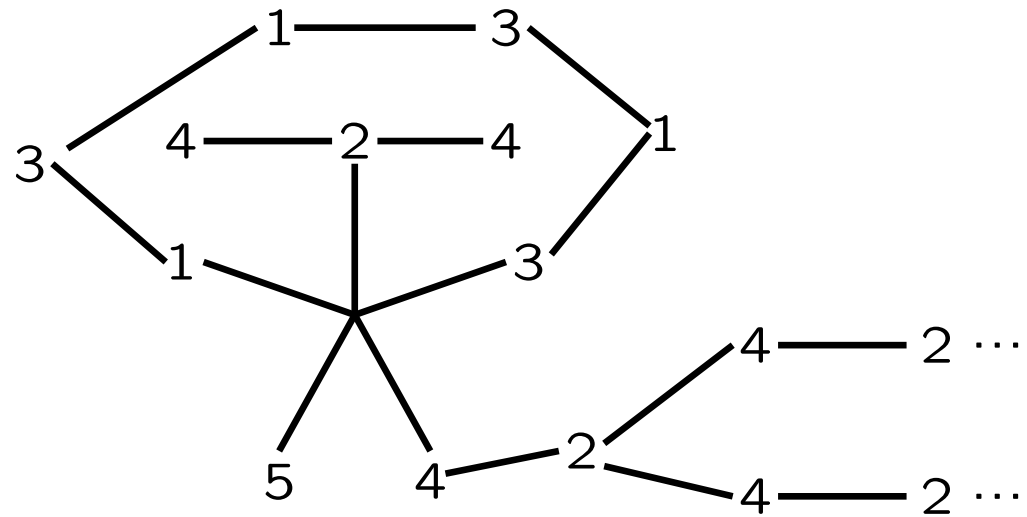


If the adjacent vertex colored 1 and the adjacent vertex colored 3 are not connected by a path in this subgraph, simply exchange the colors 1 and 3 throughout the subgraph connected to the vertex colored 1.

This will leave color 1 available to color the "special" vertex, and we are done.



On the other hand, if the vertices colored 1 and 3 are connected via a path in the subgraph, we do the same "subgraph" process with vertices colored 2 and 4 adjacent to the "special" vertex. Note that this will be a disconnected pair of subgraphs, separated by a path connecting the vertices colored 1 and 3. Now we can exchange the colors 2 and 4 in the subgraph connected to the adjacent vertex labeled 2. This will leave color 2 for the "special" vertex.



Thus, we will be able to color the entire planar graph with 5 colors, and the induction is done. Q.E.D.

The 4-Color Theorem



It turns out that it is actually a theorem that 4 colors are enough for any planar graph (map-coloring . . .). The proof of that fact is significantly more difficult, and has only been done with the aid of exhaustive computer analysis of many special cases.