A very brief introduction to differentiable manifolds

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Our general topics:

Why differentiable manifolds

- Differentiable manifolds can generally be thought of as a generalization of $\mathbb{R}^n$. They are mathematical objects equipped with smooth (local) coordinate systems. Much of physics can be thought of as having a natural home in differentiable manifolds. A particularly valuable aspect of differentiable manifolds is that unlike traditional flat (Euclidean) $\mathbb{R}^n$, they can have (intrinsic) curvature. ←

Topological spaces

- We need a way to talk about "nearness" of points in a space, and continuity of functions. We can't (yet) talk about the "distance" between pairs of points or limits of sequences – we will use a more abstract approach. We start with: ←

  Def.: A **topological space** $(\mathbf{X}, \mathbf{T})$ is a set $\mathbf{X}$ together with a topology $\mathbf{T}$ on $\mathbf{X}$. A topology on a set $\mathbf{X}$ is a collection of subsets of $\mathbf{X}$ (that is, $T \subset \mathcal{P}(\mathbf{X})$) satisfying:

  1. If $G_1, G_2 \in \mathbf{T}$, then $G_1 \cap G_2 \in \mathbf{T}$.

  2. If $\{G_\alpha \mid \alpha \in J\}$ is any collection of sets in $\mathbf{T}$, then
  $$\bigcup_{\alpha \in J} G_\alpha \in \mathbf{T}.$$

  3. $\emptyset \in \mathbf{T}$, and $\mathbf{X} \in \mathbf{T}$.

- The sets $G \in \mathbf{T}$ are called *open sets* in $\mathbf{X}$. A subset $F \subset \mathbf{X}$ whose complement is open is called a *closed set* in $\mathbf{X}$.

- If A is any subset of a topological space X, then the *interior* of A, denoted by A°, is the union of all open sets contained in A. The *closure* of A, denoted by $\overline{A}$, is the intersection of all closed sets containing A.

- If $x \in X$, then a *neighborhood* of x is any subset $A \subset X$ with $x \in A^\circ$.

- If (X, T) is a topological space, and A is a subset of X, then the *induced* or *subspace* topology T_A on A is given by

$$T_A = \{G \cap A \mid G \in T\}.$$

  It is easy to check that T_A actually is a topology on A. With this topology, A is called a *subspace* of X.

- Suppose X and Y are topological spaces, and $f : X \to Y$. Recall that if $V \subset Y$, we use the notation

  $$f^{-1}(V) = \{x \in X \mid f(x) \in V\}.$$

  We then have the definition:

  Def.: A function $f : X \to Y$ is called *continuous* if $f^{-1}(G)$ is open in X for every open set G in Y.

- We can also define continuity at a point. Suppose $f : X \to Y$, $x \in X$, and $y = f(x)$. We say that f is *continuous at x* if for every neighborhood V of y, there is a neighborhood U of x with $f(U) \subset V$. We then say that a function f is *continuous* if it is continuous at every $x \in X$.

- A *homeomorphism* from a topological space X to a topological space Y is a 1-1, onto, continuous function $f : X \to Y$ whose inverse is also continous.

- A topological space is called *separable* if there is a countable collection of open sets such that every open set in T can be written as a union of members of the countable collection.

- A topological space X is called *Hausdorff* if for every $x, y \in X$ with $x \neq y$, there are neighborhoods U and V of x and y (respectively) with $U \cap V = \emptyset$.

- This is just the barest beginnings of Topology, but it should be enough to get us off the ground . . .

Topological spaces - exercises

1. Show that the intersection of a finite number of open sets is open. Give an example to show that the intersection of an infinite number of open sets may not be open. ←

2. How many distinct topologies are there on a set containing three elements?

3. Show that the interior of a set is open. Show that the closure of a set is closed. Show that $A^\circ \subset A \subset \overline{A}$. Show that it is possible for A° to be empty even when A is not empty.

4. Show that if $f : X \to Y$ is continuous, and $F \subset Y$ is closed, then $f^{-1}(F)$ is closed in X.

5. Show that a set can be both open and closed. Show that a set can be neither open nor closed.

6. Show that if $f : X \to Y$ and $g : Y \to Z$ are both continuous, then $g \circ f : X \to Z$ is continuous.

7. Show that the two definitions of continuity are equivalent.

8. A subset $D \subset X$ is called *dense* in X if $\overline{D} = X$. Show that it is possible to have a dense subset D with $D^\circ = \emptyset$.

9. Show that if D is dense in X, then for every open set $G \subset X$, we have $G \cap D \neq \emptyset$. In particular, every neighborhood of every point in X contains points in D.

10. Show that in a Hausdorff space, every set consisting of a single point x (i.e., $\{x\}$) is a closed set.

Examples of topological spaces

- For any set X, there are two trivial topologies: ←

  $$T_c = \{\emptyset, X\}$$

  and

  $$T_d = \mathcal{P}(X).$$

  T_d is the topology in which each point (considered as a subset) is open (and hence, every subset is open). It is called the *discrete* topology. T_c is sometimes called the *concrete* topology.

- On $\mathbb{R}$, there is the usual topology. We start with open intervals $(a, b) = \{x \mid a < x < b\}$. An open set is then any set which is a union of open intervals.

- On $\mathbb{R}^n$, there is the usual topology. One way to get this is to begin with the open balls with center a and radius r, where $a \in \mathbb{R}^n$ can be any point in $\mathbb{R}^n$, and r is any positive real number:

  $$B_n(a, r) = \{x \in \mathbb{R}^n \mid |x - a| < r\}.$$

  An open set is then any set which is a union of open balls.

Examples of topological spaces - exercises

1. Check that each of the examples actually ← is a topological space.

2. For $k < n$, we can consider $\mathbb{R}^k$ to be a subset of $\mathbb{R}^n$. Show that the inherited subspace topology is the same as the usual topology.

3. Show that $\mathbb{R}^n$ with the usual topology is separable and Hausdorff.

Coordinate systems and manifolds

- Suppose M is a topological space, U is an open subset of M, and $\mu : U \to \mathbb{R}^n$. Suppose further that $\mu(U)$ is an open subset of $\mathbb{R}^n$, and that μ is a homeomorphism between U and $\mu(U)$. We call μ a *local coordinate system* of dimension n on U. $\leftarrow$

  For each point $m \in U$, we then have that $\mu(m) = (\mu_1(m), \ldots, \mu_n(m))$, the coordinates of m with respect to μ.

- Now suppose that we have another open subset V of M, and ν is a local coordinate system on V. We say that μ and ν are C^∞ *compatible* if the composite functions $\mu \circ \nu^{-1}$ and $\nu \circ \mu^{-1}$ are C^∞ functions on $\mu(U) \cap \nu(V)$. Remember that a function on $\mathbb{R}^n$ is C^∞ if it is continuous, and all its partial derivatives are also continuous.

- A *topological manifold* of dimension n is a separable Hausdorff space M such that every point in M is in the domain of a local coordinate system of dimension n. These spaces are sometimes called *locally Euclidean* spaces.

- A C^∞ *differentiable structure* on a topological manifold M is a collection $\mathcal{F}$ of local coordinate systems on M such that:

  1. The union of the domains of the local coordinate systems is all of M.

  2. If μ_1 and μ_2 are in $\mathcal{F}$, then μ_1 and μ_2 are C^∞ compatible.

  3. $\mathcal{F}$ is maximal with respect to 2. That is, if ν is C^∞ compatible with all $\mu \in \mathcal{F}$, then $\nu \in \mathcal{F}$.

- A C^∞ *differentiable manifold* of dimension n is a topological manifold M of dimension n, together with a C^∞ differentiable structure $\mathcal{F}$ on M.

  Notes:

  1. It is possible for a topological manifold to have more than one distinct differentiable structures.

  2. In this discussion, we have limited ourselves to C^∞ differentiable structures. With somewhat more work, we could define C^k structures for $k < \infty$.

  3. We have limited the domains of our local coordinate systems to be open subsets of M. This means that the usual spherical and cylindrical coordinate systems on $\mathbb{R}^3$ do not count as local coordinate systems by our definition.

4. With somewhat more work, we could define differentiable manifolds with boundaries.

5. We have limited ourselves to manifolds of finite dimension. With somewhat more work, we could define infinite dimensional differentiable manifolds.

Coordinate systems and manifolds - exercises

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Manifolds

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Manifolds - exercises

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References

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