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An out-of-equilibrium model of the distributions of wealth

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Abstract

The distribution of wealth among the members of a society is herein assumed to result from two fundamental mechanisms: trade and investment. An empirical distribution of wealth shows an abrupt change between the low–medium range, that may be fitted by a non-monotonic function with an exponential-like tail such as a gamma distribution, and the high wealth range, that is well fitted by a Pareto or inverse power-law function. We demonstrate that an appropriate trade-investment model, depending on three adjustable parameters associated with the total wealth of a society, a social differentiation among agents, and economic volatility referred to as investment, can successfully reproduce the distribution of empirical wealth data in the low, medium and high ranges. Finally, we provide an economic interpretation of the mechanisms in the model and, in particular, we discuss the difference between classical and neoclassical theories regarding the concepts of *value* and *price*. We consider the importance that out-of-equilibrium trade transactions, where the prices differ from values, have in real economic societies.

1. Introduction

In this paper we attempt to replicate the empirical distribution of wealth using a parsimonious model in which there are two ways by which wealth can be accumulated: by investment and by trading. By investment we intend any act that creates or destroys wealth, whereas by trade we intend any type of economic transaction. While a basic investment mechanism can be modelled with a stochastic multiplicative model [1], herein we focus on a basic nonlinear stochastic trade mechanism that captures the following properties:

(a) in trades there may be a transfer of wealth from one agent to the other because the price paid fluctuates around an equilibrium price (= value) and, therefore, the price may differ from the value of the commodity transferred;
(b) in a trade transaction the amount of wealth that may move from one agent to the other is bounded because the price and the value of a commodity cannot (usually) exceed the wealth of the poorer of the two traders;
(c) the price is socially determined in such a way that the trade is statistically biased in favour of the poorer trader.

These properties imply an intrinsic nonlinear stochasticity and cannot be captured by linear econophysics models such as one based on a mean-field approximation [1], or on a kinetic theory approximation [2]. We refer to our model as the nonlinear stochastic trade-investment model (NSTIM).

There is a difference between the price paid for and the value of the commodity transferred in a trade. This difference is the basis of the NSTIM and requires an economic explanation. In fact, the concept of a divergence between price and value is foreign to neoclassical economic theory, which assumes that all trades occur in 'equilibrium' [3], where price and value are equal. This economic assumption is summarized in the so-called '*law of one price*'[4]. Neoclassic economists do not expect trade to involve a transfer of wealth, but rather an increase in utility for both parties with a zero net transfer of wealth. This balance was formalized into the controversial '*Say's principle*' [4][5]. While the concepts of price and value have effectively been melded in neoclassical theory, its predecessor, the classical school of thought, maintained a distinction. Value reflected the long-term cost of production of a commodity in classical economics, whereas price was what was paid in the market on any particular day. The classical economists expected divergences to occur regularly, though they also expected that there would be a tendency for market price to converge towards value over time. We observe that the familiar expectation that a more modern theory is better than an older one may not be true in economics, where conventional neoclassical concepts, theories, and methodologies are still being criticized [5].

To physicists, given the dynamics of real economic interactions and the many social and political forces that impinge on economic interactions, the argument that actual trades occur in equilibrium and are market clearing appears strange. We expect that trades will usually be out-of-equilibrium, that is, price ought to differ from value, and the existence of fluctuating levels of stocks of all kinds of unsold goods is manifest proof that real prices are not 'market clearing'. Data show that the so-called '*law of one price*' is never fulfilled because the market price does not converge towards value even after several centuries [6]. Wealth transfer should therefore always occur in real trades because it is possible to buy or sell a commodity for a price that may be higher or lower than its value around which actual transaction prices fluctuate.

Moreover, the total amount of wealth that can be transferred in trades from one agent to another should be constrained by the total amount of wealth of the poorer trader because a trader cannot (usually) afford to buy or sell a commodity whose value or price is larger than his/her own total wealth. The interesting finding we reach is that because of the above constraint, the richer agent always risks less than does the poorer agent in trades. Therefore, if there is an equal probability to make a '*good* deal' for both the richer and poorer traders, that is, we assume that all possible outcomes within the wealth bound are equally likely, the overall effect is that the entire wealth of a society will eventually concentrate through successive random trades into the hands of the rich; this is a classical statistical problem known as the *gambler's ruin* [7]. This may explain why it is easier for the rich to get richer and the poor to get poorer over time, than is the reverse. Of course, in real societies a situation in which the entire wealth concentrates in the hands of very few people will cause an economic instability resulting in a social catastrophe. A stable society requires that the middle class be the largest.

[4] An economic rule which states that in an efficient market, a security must have a single price, no matter how that security is created. For example, if an option can be created using two different sets of underlying securities, then the total price for each would be the same or else an arbitrage opportunity would exist. About the deviation from the *'law' of one price*, see also [6].

[5] For an original critique of 'Say's law' see chapter 12, by S Keen.

Indeed, real wealth and income distributions show that the middle class is the largest. The income distributions, $p(w)$, for very large w, almost 1% of the population (the rich class), satisfy an inverse power law (IPL) of the form

$$\lim_{w\to\infty} p(w) \propto \frac{1}{w^{\mu+1}}, \tag{1}$$

where μ is the IPL index (note that μ is known as Pareto's exponent [8], of the cumulative distribution $P(w) = \int_w^\infty p(x)\,\mathrm{d}x \propto w^{-\mu}$). Instead, for the low to middle level of wealth, almost 99% of the population, the wealth distributions present a different shape. The distribution curve is non-monotonic and, therefore, there is a large middle class and a smaller poor class. Wealth or income distributions at the low–middle range may be well fitted, for example, by a gamma distribution,

$$\lim_{w\leftarrow\infty} p(w) \propto w^{\eta} \exp[-dw], \tag{2}$$

where η and d are appropriate parameters. Figure 1 shows some phenomenological distributions and cumulative distributions of wealth and income for the United Kingdom and the United States [9]. The change of properties between low–middle (gamma-distribution) and high wealth (Pareto's law) ranges is typical of these distribution [10–12].

The social economic catastrophe related to the *gambler's ruin* problem can be avoided by assuming a systematic and effectively permanent stochastic bias in trade that favours the poorer trader. This pro-poor bias would move wealth from rich to poor and reproduce the above socially safer gamma-like distribution with a large middle class and a smaller poor class. We observe that social mechanisms such as the graduated income tax and luxury tax policies are statistically biased in favour of the poor and may play significant roles in moving wealth from rich to poor. In addition, because the main trade that poor to middle wealth individuals make is of their labour services in return for a wage, we suggest that the wage tends to stochastically exceed the value of labour. Even if a reader may initially be surprised by our argument that workers' wages should be higher than the value of their labour, we recall that this phenomenon was also observed by Karl Marx. In his complete analysis Marx made clear that the wages would normally exceed the value of labour-power [13–15] because 'The natural price of labour is nothing but the minimum wage' (p 55 in [13]), which is a subsistence payment. Indeed, there should be an economic bias in favour of the workers because workers do not receive only a subsistence payment and, therefore, the wage 'price' should normally exceed wage 'value'!

In fact, the wage 'price' is not simply the product of the labour itself but also depends on other factors such as, for example, the social and political necessity of stabilizing and favouring the entire society. Moreover, for example, the capitalists would have to pay a kind of 'social peace' component to avoid strikes because workers might not be satisfied to receive only a subsistence payment. The main means for getting a wage higher than a subsistence payment is given by the strength of the poor and middle classes that

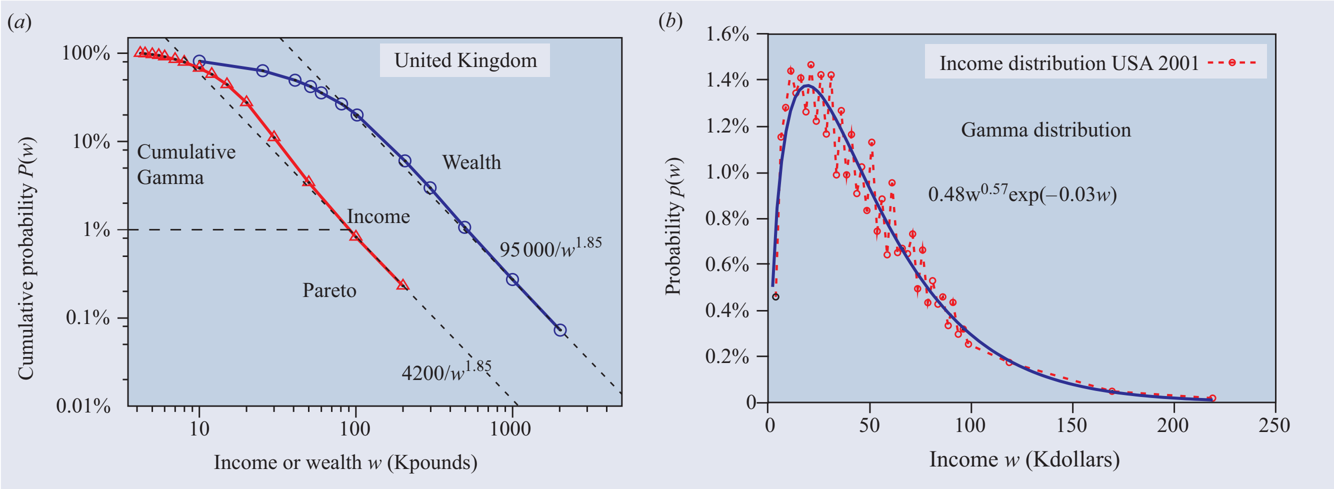

Figure 1. (a) Cumulative wealth (1996) and income (1998–9) distributions in the United Kingdom; for the upper class (almost 1% of the population) the distribution is well fitted with a Pareto's law. (b) Income distribution (2001) in the USA; for the low–middle classes (almost 99% of the population), the distribution is well fitted with a gamma distribution, equation (15).

are much larger and, in particular in modern democracies, politically more influential than the very small rich class. The social, political and psychological effect of the higher economic risk in which the low–middle classes live, due to their economic resources that are more restricted than those of the rich, is probably what ultimately determines that real prices and wages are forced to be in an out-of-equilibrium state that favours the poor–middle classes.

The NSTIM developed herein attempts to simply model the above properties. The paper has the following structure. In section 2 we present some historical background and indicate what is to be expected from a model based on complex system theory. In section 3 we introduce the nonlinear stochastic model schematizing the mechanisms of trade and investment, presumed to underlie the allocation of wealth within a society. In section 4 we proceed to analyse the NSTIM by computer simulations, under varying assumptions for the intervening parameters. In section 5 the model is shown to be capable not only of yielding outcomes which compare favourably with actual data but also of shedding some light on the economic basis for certain overall features of the data. Finally, in section 6 we present a detailed discussion of the meaning, in an economic context, of the terms appearing in the model, and then venture to draw some conclusions on the implications of the foregoing considerations for the welfare of human societies.

2. Some historical observations and asset exchange linear models

It is straightforward to determine that while income and wealth distributions look qualitatively similar (see figure 1), they do not coincide because income and wealth are not the same quantity.

The Oxford Dictionary of Economics [16] defines *wealth* as 'the total value of a person's net assets, such as, money, shares in companies, debt instruments, land, buildings, intellectual property such as patents and copyrights, and valuables such as works of art. From this any debts owed are subtracted'. The same source proceeds to indicate that 'the valuation put on these things is liable to uncertainty and fluctuations, as many of the assets are not marketed, and those that are may have volatile market prices'. The latter aspect will turn out to play a crucial role in the model. In more quantitative terms a wealth distribution can be obtained by using the data of all assets and liabilities of a person that must be reported at his or her death for the purpose of inheritance tax as Drăgulescu and Yakovenko [11] did for estimating the wealth distribution of United Kingdom. On the other hand, *income* is defined as the amount of money or its equivalent received during a period of time in exchange for labour or services, from the sale of goods or property, or as profit from financial investments. An income distribution is obtained by using data reported each year to the state for income tax purposes [11]. Intuitively, it appears that wealth and income must be somehow related; the variables wealth, income and, we add, consumption are known, in a society of interacting agents, to affect each other in a highly non-trivial manner, through complicated feedback loops, partly described by the economic concepts of 'wealth effect' and 'income effect' [16, 17].

Both wealth and income have a long history of being studied as quantitative indicators of the economic status of a society, at least since the classic work of Pareto [8], in which he proposed that the cumulative distribution of the rather sparse income data pertaining to a number of societies, available to him, could be fitted by an IPL[6]. With far better data, both in terms of quantity and quality, being available to later researchers, it is now fairly evident that income distributions have a more complicated structure, reasonably well described by IPLs only in the uppermost bracket [18]. There appears to be a consensus among various scientists [1, 11, 19, 20] that the

[6] It may be of interest to note that Pareto, in overt polemic with some of his contemporaries, believed in the primacy of empirical data over preconceived theories. He is also generally credited with having introduced power laws as relevant phenomenological distributions [8].

cumulative distribution of wealth, for the very wealthiest, also exhibits an IPL form, while the data pertaining to other social strata are less certain. An empirical PDF of wealth increases in the low range, reaches a maximum at middle range, and finally decreases as an IPL at high values of wealth [9, 11].

Economists have long sought to devise theoretical models that could reproduce empirical wealth data or, even more ambitiously, that could 'predict' how the distribution of wealth in a society would respond to changing conditions. None of the existing models are considered entirely satisfactory [21]. Such efforts have been frustrated by the previously lamented difficulty in obtaining unequivocal data, especially for the less wealthy [22, 23], but also, we believe, by an approach to the problem that tends to identify and incorporates a large array of factors that are believed to affect the dynamics of wealth [21]. Framed in such reductionistic terms, the problem clearly becomes extremely complicated.

An alternative to a direct modelling of all the variables in a system emerges from a strategy for modelling complex systems [24]. The main challenge is to extract the essence of what appears to be a forbiddingly complicated behaviour. In analogy with statistical mechanics, a number of authors have sought to derive wealth distributions as the steady-state solutions of differential equations consisting of a stochastic term (investment) and an interaction term (trade) [1, 19, 25–27]. In these models an individual's wealth may change in time either (a) because the valuation of wealth is 'liable to uncertainty and fluctuations', or (b) because wealth may be exchanged among members of a society. Consequently, we refer to (a) as *investment* and (b) as *trade*, although such terms can also be used in a broader sense in these models. A common trait shared by such models is that the effect of investment is incorporated through a stochastic multiplicative process term, which is known to yield solutions with inverse power-law behaviour [1, 27, 28]. Differences arise, however, in the interaction terms describing the effect of trade [1,29,30].

A linear model proposed by Bouchaud and Mezard [1], and also used by other authors [19,25,26], was borrowed from the physics of directed polymers, and describes the dynamics of the wealth $W_i(t)$ of the agent i in an ideal society of N agents as given by

$$\frac{\mathrm{d}W_i}{\mathrm{d}t} = \eta_i(t)W_i + \sum_{j=1(\neq i)}^{N} J_{ij}W_j - \sum_{j=1(\neq i)}^{N} J_{ji}W_i. \quad (3)$$

The component $\eta_i(t)W_i$ is a Gaussian multiplicative process with variance σ that simulates the investment dynamics. The two sum terms of equation (3) describe the trade interaction network between the agent i and all other agents in the society, and J_{ij} is the linear exchange rate between agents i and j. This model is solvable in the mean-field approximation that implies that $J_{ij} = J/N$ and yields

$$\frac{\mathrm{d}W_i}{\mathrm{d}t} = \eta_i(t)W_i + J(\overline{W} - W_i), \quad (4)$$

where $\overline{W} = N^{-1}\sum_i W_i$ is the mean wealth and J is the mean-field coefficient. The mean-field approximation is useful because equation (4) can be associated with a solvable Fokker–Planck equation with the equilibrium pdf solution $p_{\mathrm{eq}}(w) = \Psi \exp[\frac{1-\mu}{w}]\frac{1}{w^{1+\mu}}$, where $\Psi = (\mu - 1)^{\mu}/\Gamma[\mu]$ is the normalization constant and $\mu = 1 + J/\sigma^2$ is the Pareto exponent [1].

However, despite the elegance of yielding a solvable equation with a solution that succeeds in reproducing a Pareto tail, we believe that the mean-field approximation obscures any aspect of a realistic trade process. In fact, this approximation implies that in a trade the agent i gives the J percentage of his own wealth W_i to the agent j and in exchange receives the same J percentage of the jth agent's wealth W_j. This means that in any trade between rich and poor, the poor will always receive an unrealistically large amount of wealth from the rich, and this amount of wealth would increase if the trader is richer even for the cheapest commodity transferred. Moreover, in the absence of the multiplicative process, the mean-field approximation causes the wealth of all economic agents to exponentially converge toward the mean wealth $\overline{W}$. In fact, the solution of equation (4) without the multiplicative process is

$$W_i(t) = \overline{W} + (W_i(0) - \overline{W})\exp[-Jt], \quad (5)$$

implying that the trade dynamics has the asymptotic effect of 'equalizing' the wealth among all members of the society. Therefore, in the absence of investments, there would be neither rich nor poor! Moreover, even disregarding the mean-field approximation, the linear trade component of equation (3) presents the further difficulty of implying that in a trade involving any type of commodity between the same ith and jth agents, the amount of wealth moving from one agent to the other is related to the wealth of both traders and that the linear exchange rates J_{ij} and J_{ji} are fixed.

There also exists a statistical equilibrium theory of markets built on the Maxwell–Boltzmann–Gibbs statistical mechanics and kinetic theory that try to handle the intrinsic stochasticity of the trade phenomenon [2, 3, 11, 12, 31]. The kinetic theory approximation assumes that the transfer of wealth between traders is similar to the interchange of energy in random elastic collisions between particles and, therefore, yields an equilibrium wealth distribution given by a Maxwell–Boltzmann exponential distribution for wealth $p(w) \propto \mathrm{e}^{-w/T}$, where T is the 'temperature' of the market. According to such an analogy, in a trade transaction the total wealth-energy (or a fraction of it according to the reaction scheme $[W_i, W_j] \rightarrow [\gamma W_i + \varepsilon(1-\gamma)(W_i+W_j), \gamma W_j + (1-\varepsilon)(1-\gamma)(W_i+W_j)]$, [31]) of both traders should mix and then be randomly split between the two traders. More recently a kinetic model of a market with random saving propensity that yields a fixed Pareto index $\mu = 1$ was also suggested [32].

The kinetic theory approach also presents problems. In fact, Maxwell–Boltzmann–Gibbs statistical mechanics requires an assumption of 'equal likelihood' that necessitates the assumption of 'absence of information' [3, 33]. These assumptions are problematic in real trades because when rich and poor interact, for example, both agents are (usually) well aware of their reciprocal social status. This reciprocal knowledge, in addition to the higher interest that the poor have in trying to save money or to get more money when they trade with the rich, should constitute the added 'information' that makes all possible price negotiation outcomes between rich and poor 'not' equally likely. The actual distribution of prices is out-of-equilibrium compared to that based on the equal likelihood assumption of prices [3] because the information that the traders have about their reciprocal social status makes the actual distribution of prices stochastically biased or skewed in favour of the poor regardless of whether the poorer is the seller or the buyer. Moreover, because the kinetic theory assumes that all outputs of the interaction are equally likely, the conservation of the sum of the two energy-wealths being the only constraint, the poor would have an extraordinary chance to significantly increase their wealth when trading with the rich. Finally, the empirical distributions of wealth or income are not monotonic; they increase, reach a maximum and, finally, decrease as an exponential function first and then asymptotically as an inverse power-law function; see figure 1. Therefore, the empirical distribution of wealth or income cannot be recovered by the monotonically decreasing exponential distribution of Maxwell and Boltzmann that, at most, is capable of fitting only the middle range of wealth or income distributions [12].

3. A nonlinear stochastic trade-investment model

Linear models have unrealistic trade interactions because they do not preserve certain important properties of actual trades. We suggest that equation (3) be replaced with a nonlinear model that preserves the main properties of a trade mechanism [34]. A trade model has to take into account the role played by prices in mediating exchange; how prices emerge from ideal negotiations among agents that may belong to different social classes; and finally, how prices are related to the values of the assets that can be associated only with the wealth of the poorer trader. In addition, the model should generate a theoretical distribution of wealth to be compared with empirical data over the entire available range, not just the uppermost bracket where the IPL prevails or the middle range where an exponential behaviour seems to prevail [11].

To incorporate these properties of trade we suggest that in the basic trade-investment model the ith agent's wealth $W_i(t)$ evolves in time according to the discrete nonlinear stochastic equation

$$W_i(t+1) = W_i(t) + r_i\xi(t)W_i(t) + \sum_{j=1(\neq i)}^{N} w_{ij}(t). \qquad (6)$$

In this model, the investment term is still a multiplicative stochastic process with Gaussian statistics, as in the Bouchaud–Mezard model. For convenience, the investment term is here rewritten as $r_i\xi(t)W_i(t)$, where r_i, the standard deviation of the Gaussian variable $r_i\xi(t)$, will be called the *individual investment index*. The new element is the quantity $w_{ij}(t)$, a nonlinear stochastic variable that may change for each transaction and which describes the actual amount of wealth that is exchanged between the agents i and j in a trade. The specification of the trade interaction term in the nonlinear stochastic model requires great care. We observe that our model has only the purpose of explaining the phenomenon in broad outline, not in minute details; therefore, the interpretation is quite abstract, but we believe realistic.

In a trade there is a flow of wealth between the two agents only if, in the case of barter, the value of the two exchanged assets is different, or, in the case of purchase, the value of the asset is different from the price paid for it. By expressing these concepts in an equation, if the trader i is the seller and the trader j is the buyer, the quantity w_{ij} is given by

$$w_{ij} = \text{price} - \text{value}; \qquad (7)$$

instead, if the trader i is the buyer and the trader j is the seller, the quantity w_{ij} is

$$w_{ij} = \text{value} - \text{price}. \qquad (8)$$

If price and value coincide, the trade would produce only a transfer of items and money among the agents, but there would not be any transfer of wealth. If the seller (buyer) succeeds in selling (buying) an item for a price that is higher (lower) than the actual value of the item, the seller (buyer) gains wealth from the buyer (seller). Therefore, we may say that it is the dynamics of making *good or bad deals,* in a generalized sense, that continuously generates the random flux of wealth from one trader to the other in every transaction.

We observe that the above wealth transfer mechanism favours one trader and not the other, and should not be confused with the fact that trades occur only if it yields some advantage to both traders. Without an increase in utility for both traders, the parties would simply refuse to trade. These reciprocal trade advantages are related to the subjective fact that each trader gets what (money or assets) he or she needs and wants more at the occurrence of the trade, while the objective transfer of wealth described by equations (7) and (8) is related to the difference between the price paid for and the value of the asset. Also we observe that if we were interested in the exchange of money instead of wealth, we can proceed with the same above formalism but with value $= 0$.

According to equations (7), (8), at each update of the economy a random amount of wealth w_{ij} is positive (negative) if the ith agent is gaining (losing) wealth in the trade. Consequently, $w_{ij} = -w_{ji}$ and a typical trade interaction is assumed to follow the scheme

$$[W_i, W_j] \rightarrow [W_i + w_{ij}, W_j + w_{ji}] \qquad (9)$$

where the antisymmetric character of the w_{ij} insures that in a trade transaction the total wealth is conserved. Conversely, wealth may be created or destroyed only by investment.

The elements w_{ij} are assumed for simplicity to be Gaussian random variables with probability density

$$p(w_{ij}) = \frac{1}{\sigma_{ij}\sqrt{2\pi}} \exp\left[-\frac{(w_{ij} - \overline{w}_{ij})^2}{2\sigma_{ij}^2}\right] \qquad (10)$$

where $\overline{w}_{ij}$ is a potential mean wealth that can be transferred between agents i and j, and

$$\sigma_{ij} = hW_{ij} \qquad (11)$$

is the standard deviation of w_{ij}. The standard deviation of w_{ij} (11) is assumed to depend on the variable

$$W_{ij} = W_{ji} = \min[W_i, W_j] \qquad (12)$$

which implements the assumption that, in a realistic trade, the fluctuation of wealth involved in a transaction must be a proper fraction ($0 \leqslant h \leqslant 1$) of the wealth of the poorer trader. In fact, an individual cannot (usually) buy or sell something that involves an amount of wealth larger than his own total wealth. Since the standard deviation of the random variable, that is, the risk incurred in a trade interaction, is proportional to h, the latter could be interpreted as a *poverty index*, in that in a poorer society a greater fraction of one's wealth would be at stake in a typical trade. One should also note that the value of σ_{ij} depends on the current status of W_i and W_j at each update.

The variable w_{ij} of equations (7), (8) is stochastic because both the price and the value of an item may fluctuate. However, we observe that the value of an item may change only in time because, as explained above, the *value* is characterized by an agreement that involves the entire society and evolves in time according to an investment mechanism. The price of an item, on the other hand, may change not only in time but it may be dispersed in space because of its local characterization of being an outcome of a negotiation that involves only the two actual traders. For example, the price of an item may be different at different locations within the same society at a given time. An explicit model would have this spatial effect included in several dynamic equations depending on several variables. Herein, instead, we imagine that this spatial dependence of the price is described by a unique nonlinear stochastic variable. Therefore, the variable w_{ij} of equation (7) is intrinsically stochastic in nature and can be associated, for example, with a distribution-like equation (10) and not with a uniquely defined price. These assumptions of the NSTIM are incompatible with the '*law of one price*' (see footnote 4), which is a standard assumption in neoclassical economic models (including Pareto's [35]) that attempt to dynamically address the problem. On the other hand these assumptions are compatible with the classical school of thought.

To complete the model we need to specify the value of the mean wealth $\overline{w}_{ij}$ to be used in equation (10). A value $\overline{w}_{ij} = 0$ would imply a statistical equilibrium in which both traders have the same chance to make a good or bad deal. However, as we have explained in the introduction, the mean wealth $\overline{w}_{ij}$ should not be zero because of the many social and political forces influencing economies. The mean wealth $\overline{w}_{ij}$ is related to a kind of social differentiation among the members of a society because the rich and the poor trade in different ways, and this keeps the trade in an out-of-equilibrium state. We assume that the poor are constrained by their poverty to be more careful in their trades and therefore, for example, they may often look for the best price opportunity for saving money. On the other hand, because of their economic strength, the rich may be willing to purchase items regardless of cost. This is a realistic assumption in particular when the value of an item is low compared with the total wealth of the trader, that is, for example when a wealthy agent would like to buy something from a poorer agent. Of course, if the wealth involved in a trade is compatible with the agent's wealth, he or she is expected to be more prudent. In other words, a trade can take place only if the two agents, through a negotiation, reach an agreement about the price of the asset. The trade transaction has a higher probability to occur if the price is below a threshold at which the buyer is willing to buy. Because this threshold increases with the total wealth of an agent, when a wealthy agent would like to buy something from a poorer agent there is a higher probability that the transaction occurs at a higher price than when a wealthy agent would like to sell the same item to a poorer agent. This asymmetric disadvantage–advantage tends to disappear when the two traders are economically equivalent.

In conclusion, in the NSTIM the volume of wealth involved in a trade between two agents is related to the wealth of the poorer of the two traders, so we assume that $\overline{w}_{ij}$ is related to the value σ_{ij}. In this way $\overline{w}_{ij}$ is of the same magnitude as the amount of wealth involved in the trade, and it is related to the total wealth of the two traders through the nonlinear expression

$$\overline{w}_{ij} = \alpha_{ij} h W_{ij}. \qquad (13)$$

The variable α_{ij} is given, for example, by the nonlinear term that measures the out-of-equilibrium status of the trade,

$$\alpha_{ij} = f \frac{W_j - W_i}{W_i + W_j}. \qquad (14)$$

The quantity $f > 0$ is called the *social equality index*, and measures the strength of the bias in favour of the poorer trader. In fact, according to equations (10) and (14), if the wealths W_i and W_j of the two traders are almost the same, then $\alpha_{ij} \approx 0$ and both traders have an equal chance of gaining or losing wealth. If, instead, for example, $W_j \gg W_i$, we have $\alpha_{ij} \approx f$, the distribution $p(w_{ij})$ (10) is out-of-equilibrium, that is, shifted toward positive values, and the trader i, the poorer trader, has a better chance to gain wealth in the trade. We stress that with this mechanism, in trading with the rich, the poor can have only a realistic 'better chance' to improve their own wealth by an amount related to their own restricted resources, and not, as the mean-field and kinetic theory approximations would imply, have the possibility of gaining an unreasonably large amount of wealth from the rich.

4. Numerical simulations

We study the properties of the NSTIM numerically and initiate all simulations by assuming an ideal society consisting of 10^5 agents, with wealth uniformly distributed among them.

The first case we examine is with $h > 0$, $f = 0$ and $r = 0$; that is, we assume a symmetric trade-alone economy where $\overline{w}_{ij} = 0$ in (10) for all j and i. Such a choice implies that in any trade each amount of transferred wealth is equally likely to be gained or lost by any given agent. The simulation shows that under such conditions, the available wealth is rapidly concentrated in the hands of relatively few agents. Moreover, the inequality in the distribution of wealth increases with the number of iterations, as shown in figure 2. Such an outcome is not surprising; in fact, if one interprets

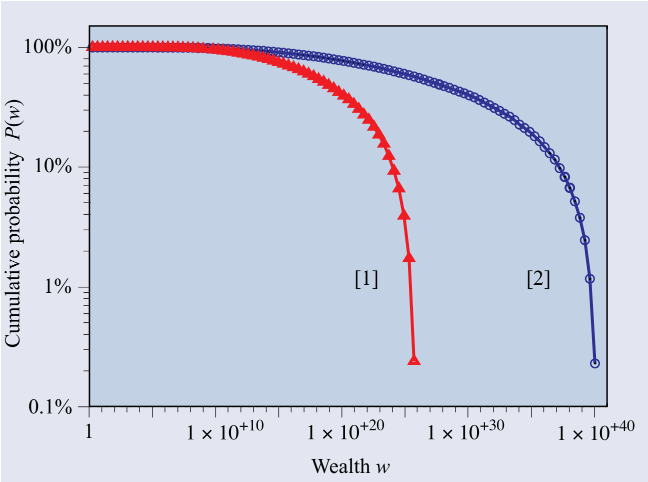

Figure 2. Cumulative wealth distribution for the symmetric trade-alone model. The indices are $h = 0.05$, $f = 0$ and $r = 0$. Case [1] is after 100 million trade-interactions, and case [2] is after 200 million trade-interactions. The initial wealth distribution is uniform. The figure shows that this model yields to a huge wealth gap between the rich and poor that increases with the number of interactions. The wealth is measured in units of the poorest agent's wealth.

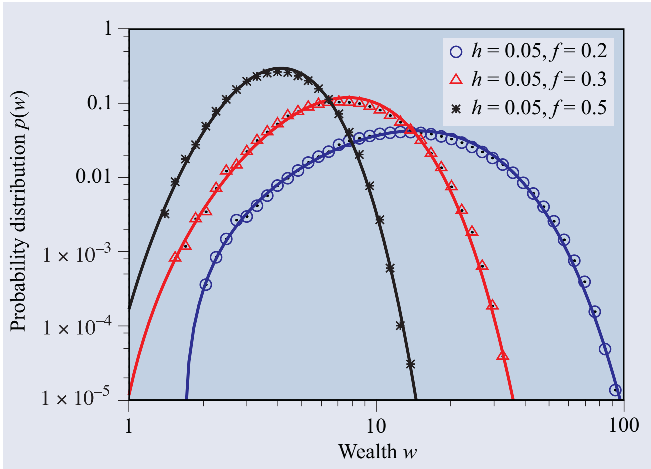

Figure 3. Wealth probability density for the asymmetric trade-alone model with a fixed poverty index $h = 0.05$. The wealth condensation increases by decreasing the social index f. The investment index is $r = 0$. The distributions are fitted by a gamma distribution equation (15). The fitting parameters are in table 1. The wealth is in units of the poorest agent's wealth.

Table 1. Fitting parameters of equation (15) for the trade-alone economy ($r = 0$); see figure 3. The poverty index is fixed: $h = 0.05$.

	a	d	c	η
$f = 0.2$	$3e{-3} \pm 3e{-4}$	$0.147 \pm 2e{-3}$	1.64 ± 0.05	1.9 ± 0.05
$f = 0.3$	$5e{-3} \pm 1e{-3}$	0.56 ± 0.02	0.8 ± 0.1	3.9 ± 0.2
$f = 0.5$	0.10 ± 0.06	1.9 ± 0.06	0.5 ± 0.1	6.8 ± 0.5

the standard deviation σ_{ij} as the *risk* incurred in a trade by agents i and j, it is clear that the poorer agent is taking a greater risk in proportion to his/her wealth, with the disparity increasing the greater the difference in wealth between the traders. The dynamics of a system such as that represented by the symmetric trade-alone model are unstable, in the sense that relative differences generally tend to be further amplified. If we set the threshold of 'economic death' at some finite level $W_j > minimum$, implying that an agent with such little wealth cannot continue to trade, the situation becomes completely analogous to the classical problem of the *gambler's ruin* [7]. Modern societies generally avert such catastrophes[7], thereby implying the unrealistic nature of trades in a statistical equilibrium between rich and poor and suggesting the need for a mechanism which would dampen differences between rich and poor, rather than amplify them.

The second case we examine is with $h > 0$, $f > 0$ and $r = 0$. The quantity $f > 0$ measures the bias in favour of the poorer trader. This version of the trade-investment model clearly implies that agent i has a greater chance to draw a positive value for w_{ij}, representing a gain, if agent i is poorer than agent j, and this chance increases with increasing economic disparity. In figure 3 we show the resulting probability density functions for different values of the social equality index f for a given value of h. The distribution of wealth becomes narrower with increasing f, meaning that the probability of being rich falls off rapidly with increasing wealth. Therefore, the condensation of wealth is less pronounced in a society characterized by a high value of the social equality index f that would imply a higher chance for the poor to gain wealth from the rich. Instead, figure 4 shows that for a given value of f, increasing the poverty index h leads to greater economic disparity and, therefore, to a higher condensation of wealth. The interpretation is that in a poorer society, and one based on trade alone, the potentially ruinous effects of an unfavourable trading sequence are more pronounced, and this will benefit the rich against the poor. All simulations shown in figures 3 and 4 are well fitted by the gamma-like distribution

$$p(w) = a(w - c)^{\eta} \exp[-dw] \qquad (15)$$

which clearly exhibits a non-monotonic behaviour with an exponential-like tail. Thus in table 1 we display the values of the parameters utilized to fit the outcomes of different simulations. To summarize, the latter version of the asymmetric or out-of-equilibrium trade-alone model ($h > 0$ and $f > 0$, $r = 0$) appears more realistic than the previous one with $f = 0$ in two respects:

(1) it yields, at least qualitatively, the emergence of a sizeable 'middle-class' as opposed to an overwhelmingly rich elite and a vast population of paupers as did the previous model;
(2) it admits well-behaved stationary solutions, whereas $f = 0$ would lead to an inexorable drift towards a singularity;
(3) the trade mechanisms give origin to a gamma distribution that is what is observed for the low–middle classes in phenomenological data, as figure 1 shows.

In fact, the economy of the low–middle classes is presumably characterized mostly by trades.

[7] Even in societies generally regarded as utterly impoverished, wealth can still be relatively abundant. This paradox is identified and discussed by De Soto [22].

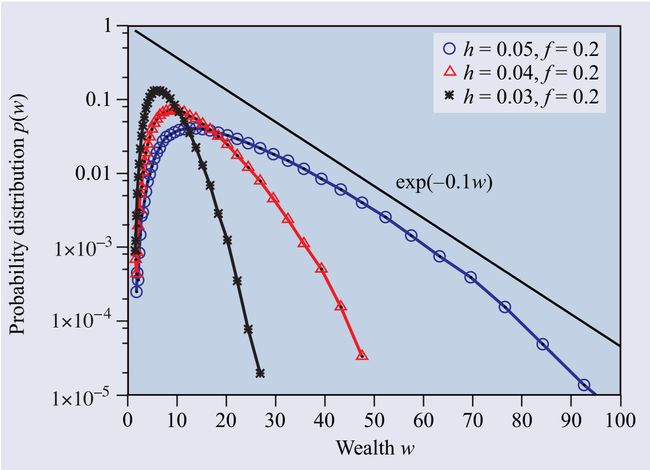

Figure 4. Wealth probability density for the asymmetric trade-alone model with a fixed social index $f = 0.2$. The wealth condensation increases by increasing the poverty index h. The distributions are compared to an exponential Maxwell–Boltzmann distribution. The wealth is in units of the poorest agent's wealth.

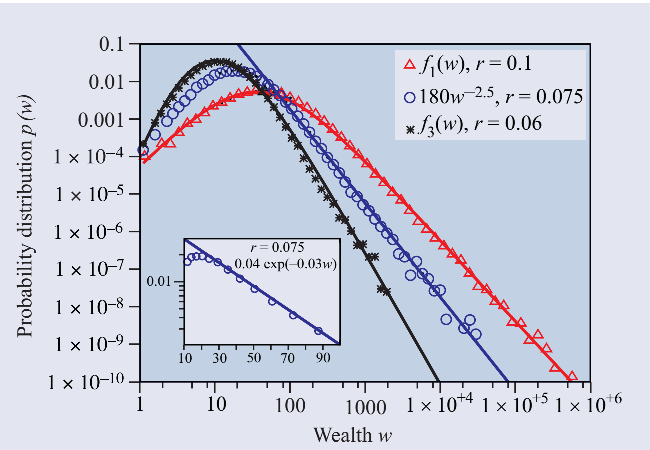

Figure 5. Trade-investment economy. The social and poverty index are fixed: $f = 0.3$ and $h = 0.05$. The probability distributions (triangles) and (stars) are fitted by using equation (16). The fitting parameters are in table 2. The tail of the probability distribution (circles) is fitted by a power law of the type $1/x^{\mu+1}$, where $\mu = 1.5$ is Pareto's exponent. The small picture shows the probability distribution (circles) in the interval [10:100] that can be apparently fitted with an exponential Maxwell–Boltzmann distribution but not at low values of w. The wealth is measured in units of the poorest agent's wealth.

Table 2. Fitting parameters of equation (16) for the trade-investment economy; see figure 5. The social and poverty indices are fixed: $f = 0.3$ and $h = 0.05$.

	a	b	γ	δ
$f1(w)$	$6e{-}5 \pm 1e{-}5$	$2.6e{-}2 \pm 2e{-}3$	2 ± 0.1	2.15 ± 0.05
$f3(w)$	$5e{-}4 \pm 1e{-}4$	0.17 ± 0.02	5.8 ± 0.6	3.5 ± 0.1

The third case we examine is with $h > 0$, $f > 0$ and $r > 0$, which means we now examine the role played by the multiplicative processes that describe investments in the model economy. In figure 5 we show the outcome of simulations

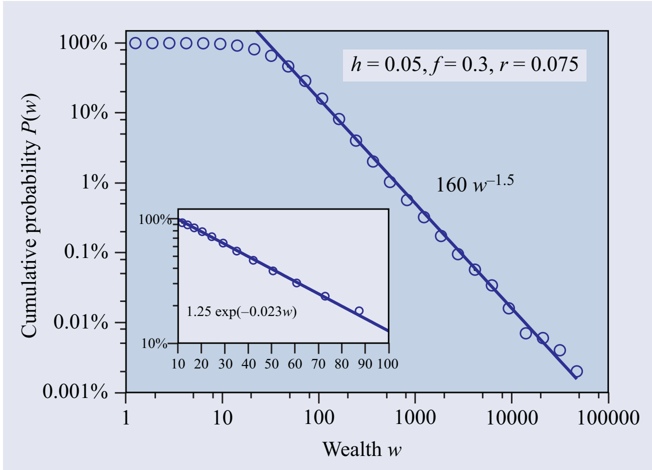

Figure 6. Cumulative probability for a trade-investment economy with $h = 0.05$, $f = 0.3$ and $r = 0.075$. Pareto's exponent is $\mu = 1.5 \pm 0.02$. The inset shows that in the interval [10:100] the $P(w)$ can be apparently fitted by an exponential Maxwell–Boltzmann distribution. The wealth is measured in units of the poorest agent's wealth.

with both mechanisms, investment and trade, simultaneously active. The curves highlight the effect of investment by plotting three probability density curves, one for each value of r_j, for fixed values of f and h. It appears that increasing r, all other things being equal, leads to a broadening of the wealth distributions, and thus to a greater economic disparity. The economic interpretation is that greater disparities arise in a society wherein one's worth tends to fluctuate more, either because its members deliberately place more of their wealth in riskier assets (investments with greater potential payoff engender more risk) or because the valuation of their assets is intrinsically more erratic due, for example, to uncertain times. As in the previous case, if we keep r fixed, the model simulations show that the Pareto index increases with increasing f and decreases with increasing h.

The investment process is implemented in the simulation by imposing that every 10^4 trades the wealth of all agents is reinitialized by the multiplicative process. In figure 5 a partial fit of the tail of one of the computer-generated probability density functions with a Pareto distribution $x^{-\delta}$, where $\mu = \delta - 1$ is the Pareto exponent, is visible. It is evident from the figure that, as expected, the inclusion of the multiplicative process term is capable of generating distributions endowed with inverse power-law tails. We find that the computer-generated probability density functions may apparently be well fitted over the entire range by functions of the form

$$p(w) = \frac{a\delta w^{\gamma}}{(1+bw)^{\delta+\gamma}}. \tag{16}$$

Table 2 records the fitting parameters for the two wealth distributions denoted by triangles and stars in figure 5. The cumulative probability function, that is, the integral of the probability density function depicted in figure 5 by circles, is shown in figure 6, along with an IPL fit for the upper wealth region. In the inset is shown the fit of the middle range of the

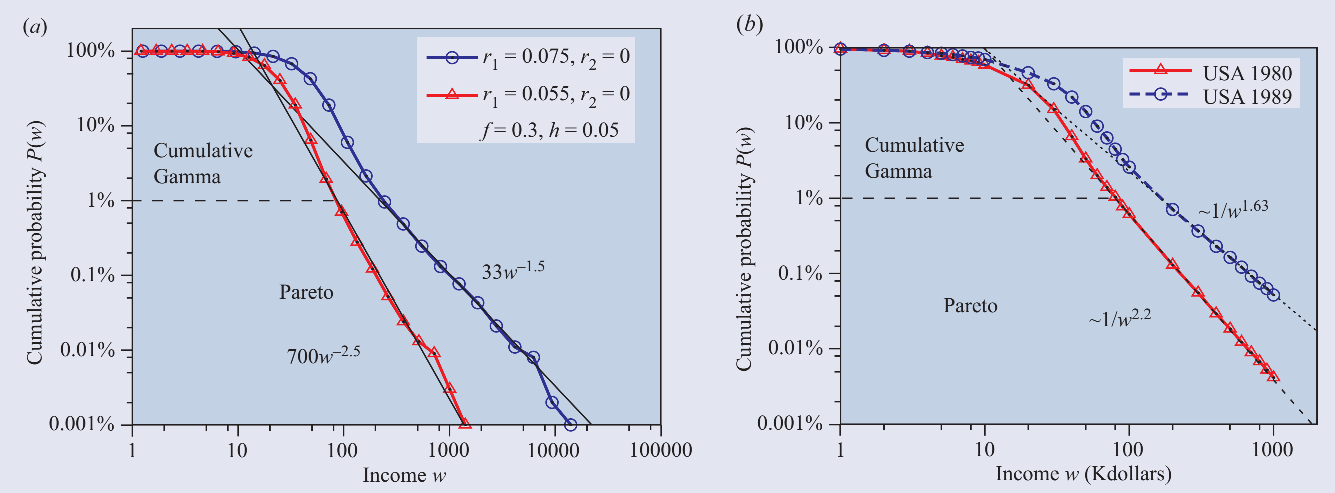

Figure 7. (a) Cumulative probability for a double trade-investment economy. One half of the population has the investment index $r_2 = 0$, the other half of the population has in one case $r_1 = 0.075$ and in the other $r_1 = 0.055$. In both cases the social index is $f = 0.3$ and the poverty index is $h = 0.05$. Pareto's exponents are $\mu = 1.5 \pm 0.02$ and 2.5 ± 0.02. 99% of the population follows a cumulative gamma distribution and 1% of the population follows a Pareto law. The wealth is measured in units of the poorest agent's wealth. (b) Cumulative income distributions in the USA during 1980 and 1989.

same curve implemented with an exponential function, proving that the low–middle range of the distribution can be also well fitted with a gamma distribution. In fact, in some simulations the distribution separation between the low–middle range and the upper range emerges more clearly.

At this stage, one can conclude that the NTSTI model (6), implemented with both investment and trade mechanisms, and with the latter biased in favour of the less wealthy agent, can successfully reproduce important qualitative features observed in the real world, namely, the stratification of society into a poor class, a large middle class and an affluent but very small upper class. A trade-alone mechanism would be sufficient to get such a stratification; it leads to a gamma distribution as observed in phenomenological data regarding the low and middle classes; see figure 1(b). To obtain the observed Pareto's law, that is, an IPL tail for the uppermost class, it is necessary to include the investment mechanism.

5. Comparison of calculations with data

In this section we assess the ability of the NSTIM to reproduce quantitative details of empirical data, see figure 1, as opposed to merely reproducing general qualitative features of data. In a recent empirical study, Drăgulescu and Yakovenko [11] conclude that the data pertaining to the cumulative distribution of wealth in the United Kingdom and income for the United States can be well fitted by an exponential (Maxwell–Boltzmann) function in the low–middle range, followed by an inverse power-law (Pareto) tail in the high-end region, as we also obtained from the preceding simulations. However, as these authors show, such an exponential behaviour cannot be correct over the entire domain of the data because the PDF of wealth is not monotonic. Drăgulescu and Yakovenko recognize that in the low–middle range the wealth PDF has been well fitted by a gamma distribution, as the NSTIM produces. In any case, it turns out that a cumulative distribution of wealth has been obtained by appeal to a variety of mechanisms. For example, Huang and Solomon [36] ascribe the appearance of such dual behaviour in wealth distributions to the fact that an idealized marketplace has a finite size; Souma *et al* [19] identify the source of these features with the rewiring of links in a small-world network [37] over which a stochastic process is assumed to evolve, and finally Montroll and Shlesinger [38], in their discussion of income, arrive at this result using renormalization group scaling arguments, and these arguments can be simply recast in terms of wealth.

To understand why the distribution of wealth for the rich is different from that of the rest of society, we observe that they rely on different economic instruments: the rich mainly on investments and all the others mainly on trades. This plausible statement has the virtue of empirical evidence [39]. A simple way to incorporate this dichotomy into the NSTIM is to subdivide the N agents into two groups with different investment indices,

$$\Delta W_i = r_1 \xi W_i + \sum_{j=1(\neq i)}^{N} w_{ij}; \qquad 1 \leqslant i \leqslant N_1 \qquad (17)$$

$$\Delta W_i = r_2 \xi W_i + \sum_{j=1(\neq i)}^{N} w_{ij}; \qquad N_1 \leqslant i \leqslant N \qquad (18)$$

where $N = N_1 + N_2$ and $r_1 \neq r_2$. Figure 7 shows two cumulative distributions obtained with the parameters social equality index $f = 0.3$, poverty index $h = 0.05$ and investment indices for 50% of the population $r_1 = 0.075$ and 0.055, while $r = 0$ for the other half in both simulations. It is clearly visible in the figure that this artificial two-tiered subdivision of society accentuates the distinction between an IPL tail region (about 1% of the population) of the wealth distribution and another region (about 99% of the population) with completely different curvature that can be fitted with a cumulative gamma distribution. Figure 7 shows that by choosing opportune parameters of the model it is possible to fit phenomenological distributions quite well. In summary, the distributions of wealth and income show an anomalous shape well described by a gamma distribution at low and middle wealth and a Pareto's tail at high wealth, that is, with a complex shape that may be approximately described by equations (1) and (2).

We interpret this result as a signal that such a duality of economic mechanisms, pursued by different strata of society, may indeed be responsible for the observed dual behaviour of empirical curves. The Pareto exponents of the two fitting curves in figure 7(a) are, respectively, $\alpha = 1.5 \pm 0.02$ and 2.5 ± 0.02, which straddle the empirical values as those shown in figure 1(a) ($\alpha = 1.85$) about the UK [11] and those shown in figure 7(b) about the cumulative distributions in the USA during 1980 ($\alpha = 2.2$) and 1989 ($\alpha = 1.63$). Furthermore, increasing r for the investment-prone part of society leads to smaller Pareto exponents, as expected, and thus to greater economic disparity.

6. Discussion and conclusions

Naturally, it would be an arduous task to measure the values of the fundamental parameters r, h and f of the NSTIM model in real economic situations. The actual values of these parameters are expected to fluctuate around average values, and they may assume arbitrarily complicated functional forms to account for varying structures of society's trade and investment networks. Nevertheless, from a complex systems viewpoint, a framework has been constructed wherein general features observed in empirical wealth distributions can be ascribed to specific activities that are known to be present in society and, perhaps more importantly, tests can be run on the social impact of variations, which are known to occur, of specific economic factors.

Phenomenological distributions of wealth show an abrupt change between the low–middle classes (99% of the population), where a gamma distribution dominates, and the uppermost class (1% of the population) where a Pareto's distribution dominates. We have shown that the NSTIM model consistently interprets such behaviour as due to the fact that trade is the main mechanism that characterizes the low and middle classes, and this produces a gamma distribution of wealth, whereas investment is the main mechanism that influences the uppermost class individuals who own the means of large production. The trade mechanism depends on two parameters: a poverty index h, and a social equality index f. We saw that by increasing h or decreasing f, wealth condenses more easily. The investment process can be modelled with a multiplicative stochastic mechanism whose strength is measured by the parameter r and generates a Pareto's law. By increasing the parameter r, wealth condenses more easily because the economy of the low–middle classes remains mainly characterized by trades even though some investment can be present within these classes. Of course, the three parameters may change in time, and this explains why the Pareto index is not constant but evolves [10] as figure 7(b) shows.

As a simple interpretative application of the model, herein we attempt an explanation of a sociological phenomenon that is easily observed in history and that was summarized by Pareto in the theory known as '*the circulation of the elites*' [40] and in his criticism of the Marxist revolutionary theory [41]. In fact, in poor societies where modern means of production are for whatever reason absent, or simply insufficiently capable of producing surplus value, revolutionaries usually think that alternate means should be developed to redistribute wealth by force or by deceit, with the results of increasing the social equality index f. But the success of such policies would be doubtful. In fact, a society without sufficient means of production would be characterized by a high poverty index h that statistically favours a small rich class in their trade with the poor. On the other hand, the social equality index f cannot be arbitrarily increased because too high an f would excessively reduce the capital that is necessary for production, and would cause a further impoverishment of the society. Moreover, the ensuing insecurity of a revolution would be reflected by a concomitant increase of the volatility parameter r, which would again favour a small upper class. Therefore, the NSTIM model seems to confirm the realistic expectation that without consistent economic development, which reduces the poverty index h that contrasts the wealth condensation, a revolution would simply have, at most, the asymptotic effect of supplanting one ruling elite with another, because there would not be a significant redistribution of wealth, despite Marxist promises to the contrary [41].

Finally, we owe the reader a wider discussion about the wisdom of opting for a model of trade that favours the poor against the rich, even if such a bias is only statistical. Without such an out-of-equilibrium bias, that is, in a truly free market where the rich and the poor have equal probability to make a good deal and the prices of all commodities symmetrically fluctuate around their own ideal value, the rich would get richer and the poor would get poorer over time; a situation that may yield deleterious social effects. The mechanism that in a trade favours the poorer of the two agents is possible because in a real economy, contrary to an assumption of the neoclassical school of economics, price and value of a commodity do not coincide: this is what generates the transfer of wealth from one trader to another according to equations (7) and (8). The price dispersion is contingent on the trade phenomenon itself, and not simply a transition to an equilibrium price as the '*law of one price*' would suggest (see footnote 4). In fact, the price dispersion is produced by different possible outcomes of economic negotiations between pairs of traders and it is associated with several factors, among them the most important being the economic differentiation among the social classes yielding a price discrimination mechanism [42][8] and therefore

[8] The *price discrimination in monopoly theory* says that a seller with a degree of monopoly power has the ability to price discriminate. This means being able to charge a different price to different customers. In fact, for example, theatres often charge youngsters or students less than others; grocery stores have lower prices for people who bother to check sale periods and look for clip coupons; some companies produce almost similar products but try to promote one as a prestige brand supposed for wealthy people at a much higher price; and so on.

a divergence between price and value, the latter defined as the equilibrium price.

The social differentiation induces asymmetric behaviours that yield statistically opposite outcomes when two agents, belonging to different social classes, negotiate a transaction; for example, the poor try to sell their services and products at the highest prices and purchase at the lowest prices. By including such a mechanism, the nonlinear stochastic trade model takes into account the importance of the role of prices in mediating exchange and redistributing wealth among members of the society. The NSTIM also takes into account how the price emerges from a negotiation mechanism whose stochastic outcome can be only a fraction of the wealth of the poorer party and cannot be related to the richer trader's wealth. This is because a trader cannot (usually) afford to buy or sell a commodity whose value is larger than his/her own total wealth. Instead, the probability of making a 'good deal' depends nonlinearly on the wealth of the two traders, and is statistically biased in favour of the poorer agent. In fact, the richer party is less risk averse when bargaining over a given amount than is the poorer party. This means that the poorer party should be a stronger bargainer than the rich to get the better of the deal.

In particular, in real societies the main trade transactions are those taking place between citizens and the state through the tax system, and between employers and employees. It is easy to realize that these processes are statistically biased in favour of the poor and play a significant role in redistributing wealth. The graduated income tax requires that the rich pay a higher percentage of taxes than do the poor. There are also luxury taxes for products that only the rich can afford to buy. In contrast, there are several tax reductions for products necessary to everybody such as, for example, food staples. These mechanisms are devices deliberately put in place to ameliorate the economic gap between rich and poor. In fact, the tax system can be interpreted as a trade through which the citizens buy services from the state. Because the services granted by the state are expected to be proportional to a citizen's wealth, the graduated tax system has the effect of discriminating among the citizens, forcing the wealthy class to pay a higher price for services than that paid by the less wealthy. About the relation between employers and employees, we notice that every time Marx spoke of the relationship between the wage and the value of labour-power, he used the term 'minimum wage', that is, a subsistence payment [13–15], thus emphasizing that in practice he expected the wage to exceed this minimum and hence there to be a price-value divergence in favour of the working class at the expense of capitalists. These effects can be incorporated into the social equality index f of equation (14) that measures the statistical bias of the trade in favour of the poor.

We observe that, according to the perspective of the neoclassical school of economics, the out-of-equilibrium concept that wage is different from the value of labour makes no sense. The neoclassical model of the labour market presumes that the wage equals the marginal product of labour (see Keen 2002 chapter 5 [5]), and wage value and wage price therefore coincide. However, from a classical point of view, the concept is quite different. The classical school defined the value of any commodity as its cost of production, and the cost of production of the commodity labour is therefore the means of subsistence. Our finding that trade is stochastically biased in favour of the poor is consistent with the fact that the wage price can be systematically higher than the means of subsistence and, therefore, the workers receive more than a subsistence wage. This empirical conclusion is not inconsistent with a neoclassical perspective on the relationship between the actual wage and the means of subsistence, since in the neoclassical theory there is no relationship between the marginal product of labour and a subsistence level of income. But the conclusion that there is a transfer of wealth from rich (employers) to poor (workers) via worker-biased wages is inconsistent with the vision of the labour market (and, therefore, all other markets) being in equilibrium.

In conclusion, we believe that the most important theoretical aspect of our findings is that our study allows us to compare two rival economic theories (classical against neoclassical) of price and wealth distribution, and seems to come down on the side of the classical school concerning the importance of the role played by price/value divergence. This divergence is alien to neoclassical economics, which has melded price and value into one thing. Neoclassical analysis is predicated upon equilibrium, and in equilibrium price equals value, and no transfer of wealth is possible in trades. The preceding classical school did have a theory in which market price could and normally did differ from value as seems phenomenologically evident. The Classical School defined value effectively as the cost of production of a commodity; the price was what it sold for on the market. They expected price/value divergences to occur all the time and that, therefore, out-of-equilibrium trades were possible. Our findings stress the importance of out-of-equilibrium mechanisms that bias the trade transactions in favour of the poor because they are not only possible but they seem necessary to stabilize society by avoiding the economic catastrophe whereby the entire wealth concentrates in the hands of very few rich people.

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