Exponentials

Tom Carter

Computer Science
CSU Stanislaus

http://csustan.csustan.edu/~tom/Lecture-Notes/Exponential/Exponential.pdf

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Our general topics:

Starting with interest ←

We'll start by looking at (compound) interest. Suppose we are a bank, offering "savings accounts" to our customers. We offer savings accounts that pay interest at an annual rate r, so that if someone puts P dollars (the "principal") in their account at the beginning of the year, then at the end of the year they will have

$$P_{new} = P_{orig} * (1 + r)$$

in their account.

Now, suppose that a customer who put P dollars in their account at the beginning of the year comes in after 6 months ($\frac{1}{2}$ year), and wants to withdraw their money "plus interest." How much should we give them? If we are a mean bank (or don't care about keeping our customers), we will say that since they didn't leave their money in the account for the entire year, we don't owe them any interest yet, and will give them back their P dollars. On the other hand, we may say, "Well, we got to use their money for half a year, so we should give them their account earnings for the half year," so we should give them

$$P_{new} = P_{orig} * (1 + \frac{r}{2})$$

But once we do that, a clever customer will come in after six months, withdraw their money plus interest, and they turn right around and deposit their money back in the account! Now, how much money will be in this customer's account at the end of the year (we're assuming that at the end of the year – Dec. 31 – we credit every customer's account with the interest their account has earned for the year). This clever customer will have an account balance of

$$P_{new} = P_{orig} * (1 + \frac{r}{2}) * (1 + \frac{r}{2}) = P_{orig} * (1 + \frac{r}{2})^2.$$

Well, before long word will get out, and all customers will want to take advantage of this opportunity for increased earnings, and all our customers will be coming in every six months, withdrawing their money and re-depositing it. Then, for their (and our) convenience, we will offer this as an "automatic" process. In other words, we will "compound semi-annually." Not only that, we will also be able to gain an advertising competitive advantage over the bank down the street. In particular, we can quote an "effective interest rate" in our advertisements.

So, for example, (in a mythical world . . . :-) if we are paying 6% interest compounded semi-annually, then we can tell our customers that our "effective interest rate" is 6.09%, using the calculation:

$$\begin{aligned}\left(1+\frac{0.06}{2}\right)^2 &= (1+0.03)^2 \\ &= 1+2*0.03+0.03^2 \\ &= 1+0.06+0.0009 \\ &= 1+0.0609\end{aligned}$$

Of course, now some clever customer will start coming in every 3 months instead, so the bank will start compounding quarterly:

$$\begin{aligned}\left(1+\frac{0.06}{4}\right)^4 &= (1+0.015)^4 \\ &= 1+0.06136\end{aligned}$$

or even weekly:

$$\left(1 + \frac{0.06}{52}\right)^{52} = 1.0617998$$

Now, obviously, your boss will ask you what the limit of this process is. What if we compound "continuously"?

$$P_{new} = P_{orig} * \lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^{n}$$

where r is the nominal annual interest rate.

So now we want to evaluate

$$\lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^{n}.$$

We do this using a standard exponential/log trick:

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^{\ln\left(\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n\right)}$$
$$= e^{\lim_{n\to\infty}\ln\left(1+\frac{r}{n}\right)^n}$$
$$= e^{\lim_{n\to\infty} n*\ln\left(1+\frac{r}{n}\right)}.$$

Now we will evaluate:

$$\lim_{n\to\infty} n*\ln\left(1+\frac{r}{n}\right) = \lim_{x\to 0}\frac{1}{x}*\ln(1+r*x)$$
$$= \lim_{x\to 0}\frac{\ln(1+r*x)}{x}$$

Then we use L'Hôpital's rule (0/0 form . . .):

$$\lim_{x\to 0}\frac{\ln(1+r*x)}{x} = \lim_{x\to 0}\frac{\frac{1}{1+r*x}*r}{1}$$
$$= r.$$

Thus, putting all the pieces together, we have that, under continuous compounding at nominal interest rate r, at the end of one year, the account will have

$$P_{new} = P_{orig} * e^r$$

and the effective interest rate will be

$$r_{eff} = e^r - 1.$$

With a bit more thinking, we can see that if a customer wants to withdraw their money after time t (in units of years), they should get:

$$P_{new}(t) = P_{orig} * e^{rt}.$$

Interest - exercises $\leftarrow$

1. We have played somewhat fast and loose with *limits* and *exponentials* and *logs*. Were all the steps legitimate?

2. Why did I shift from $\lim_{n\to\infty}$ to $\lim_{x\to 0}$ in the middle of my calculation?

3. Could we have come to the same conclusion (exponentials) in some other way (e.g., solving a differential equation)?

4. More metaphysically, why does the *exponential* show up here?