Some Fractals and Fractal Dimensions

- The Cantor set: we take a line segment, and remove the middle third. For each remaining piece, we again remove the middle third, and continue indefinitely.

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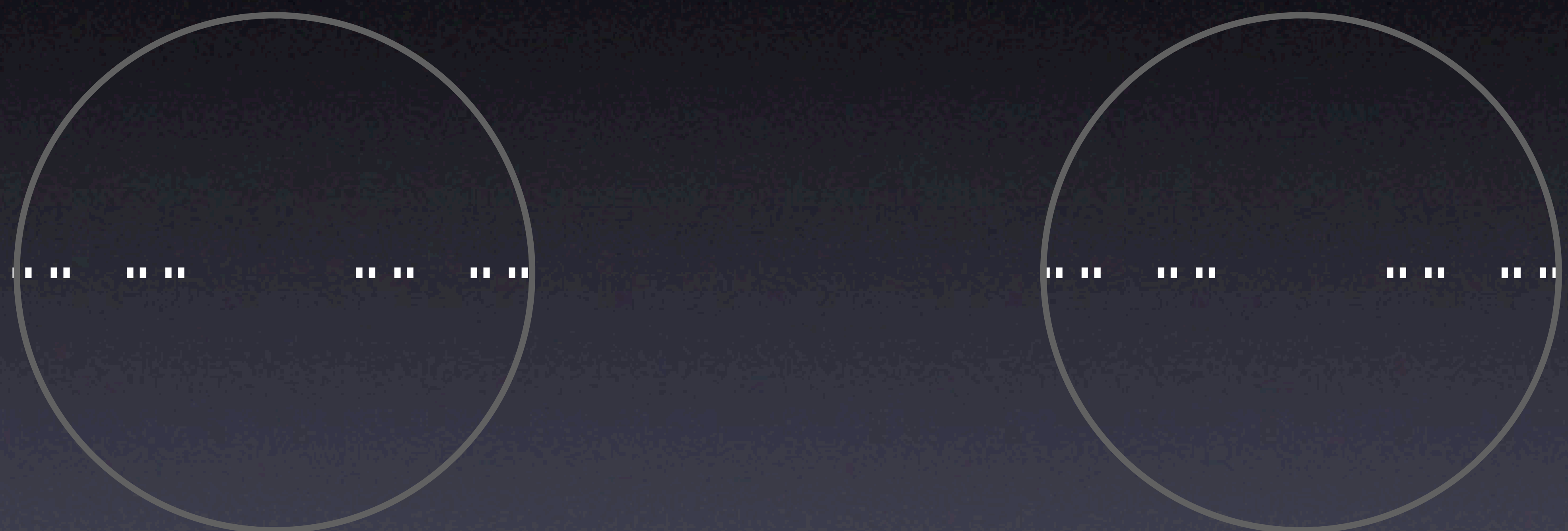
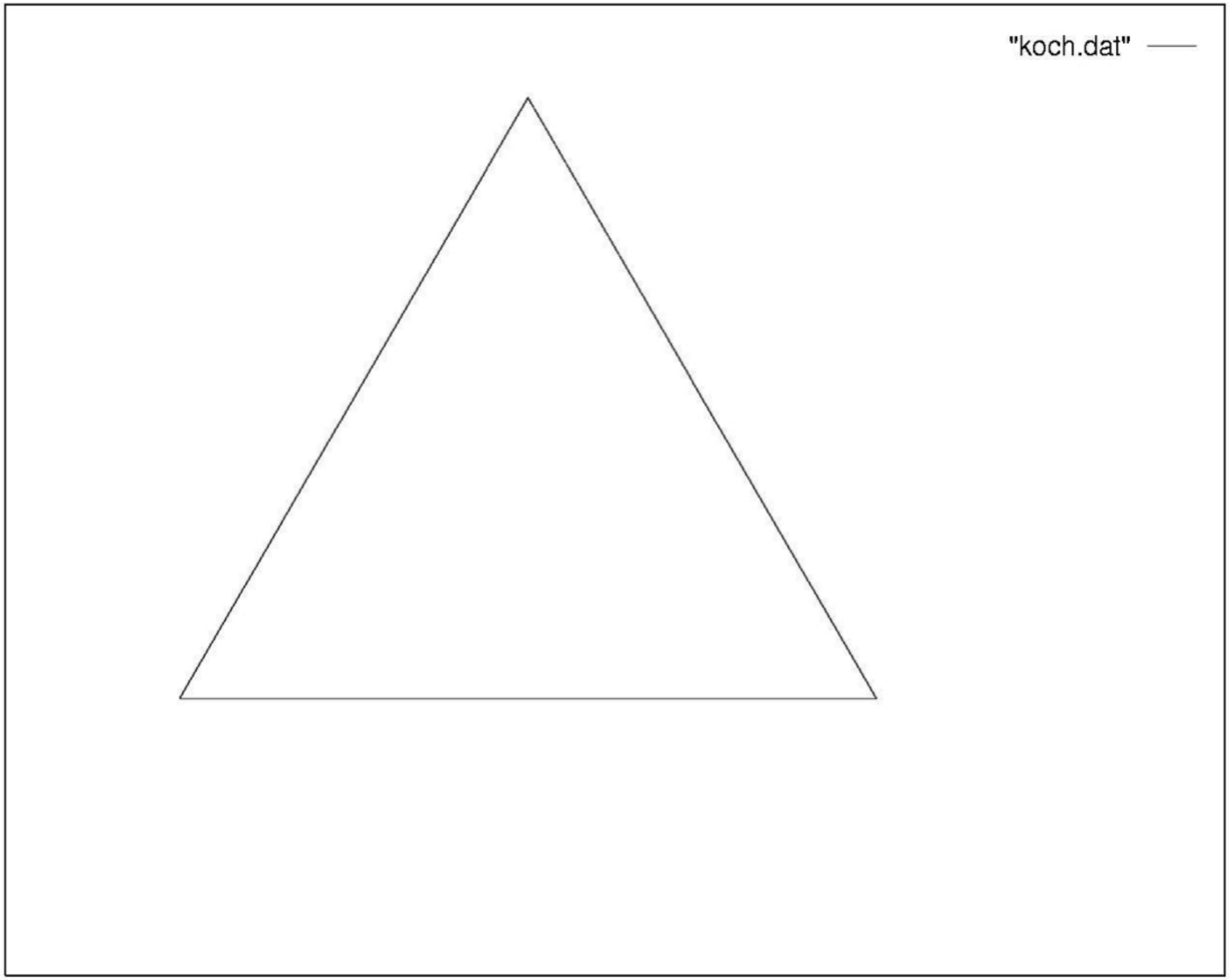
- To calculate the fractal / Hausdorff / capacity / box-counting dimension, we see how many boxes (circles) of diameter 1/r^n we need to cover the set (in this case, we will use r = 3, since it fits nicely).



$$D = \mathrm{Lim}(\log(N_r)/\log(1/r))$$
$$= \log(2) / \log(3)$$

r	N_r
1	1
1/3	2
1/3^2	2^2
1/3^3	2^3
1/3^n	2^n

- The Koch snowflake: We start with an equilateral triangle. We duplicate the middle third of each side, forming a smaller equilateral triangle. We repeat the process.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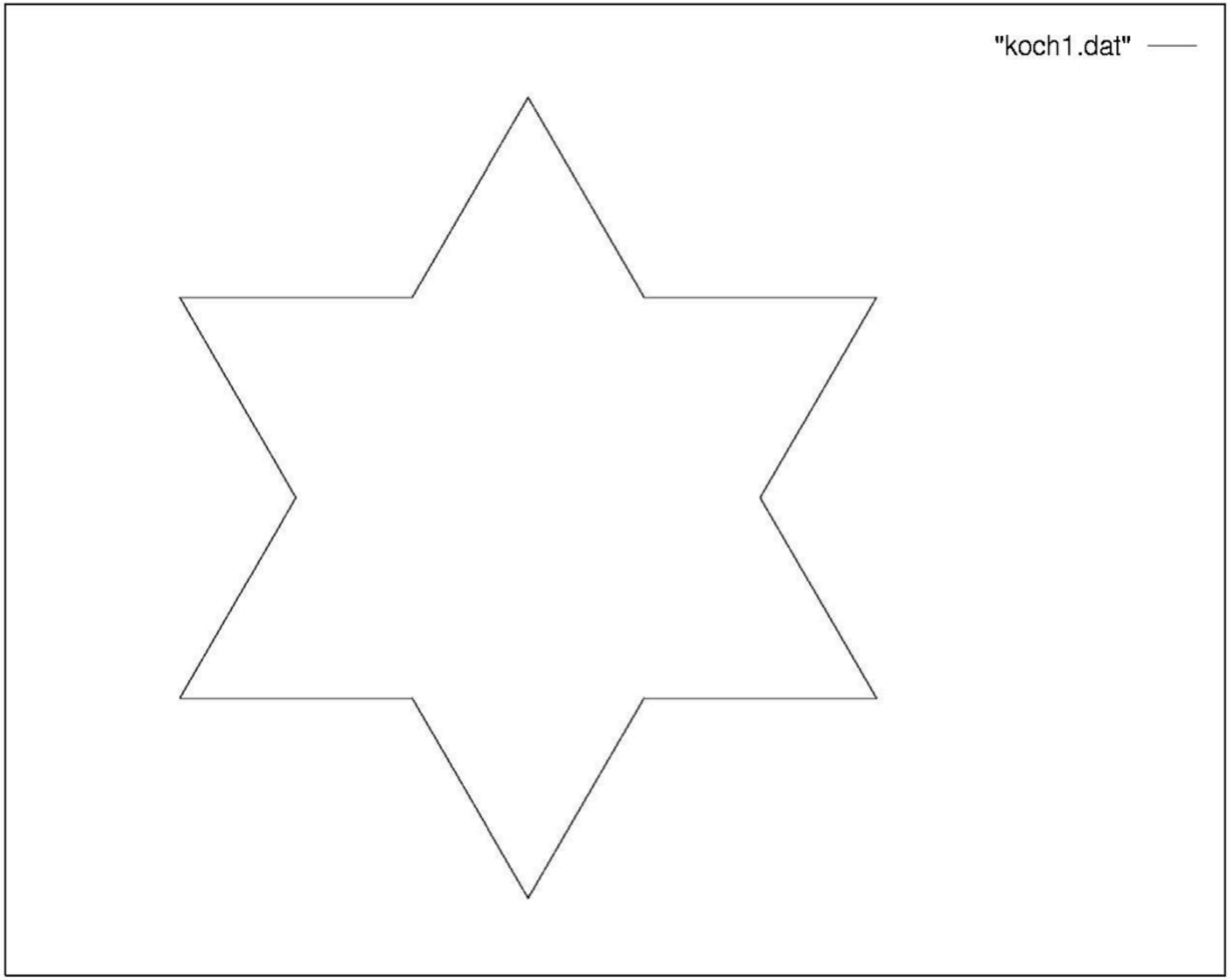
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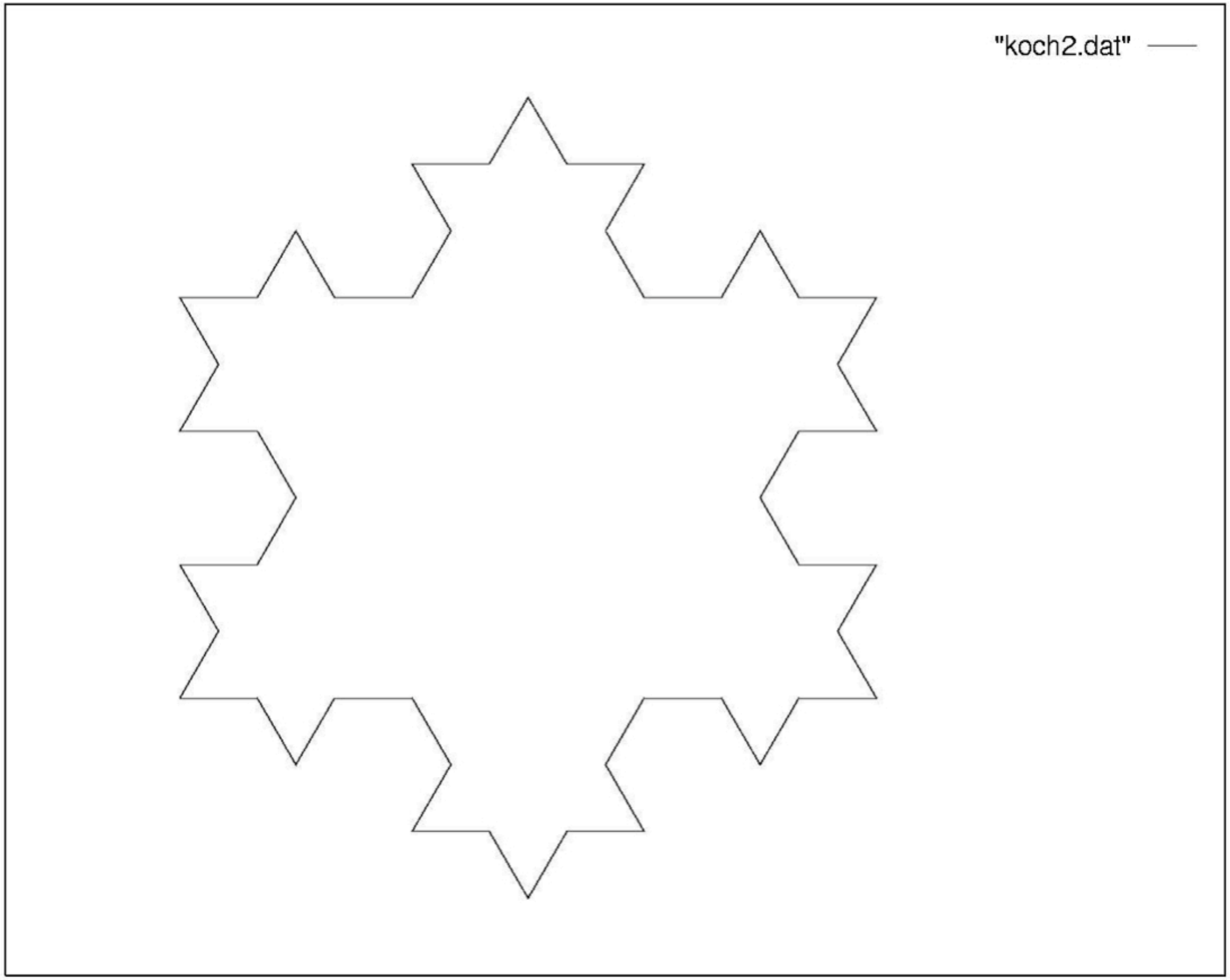
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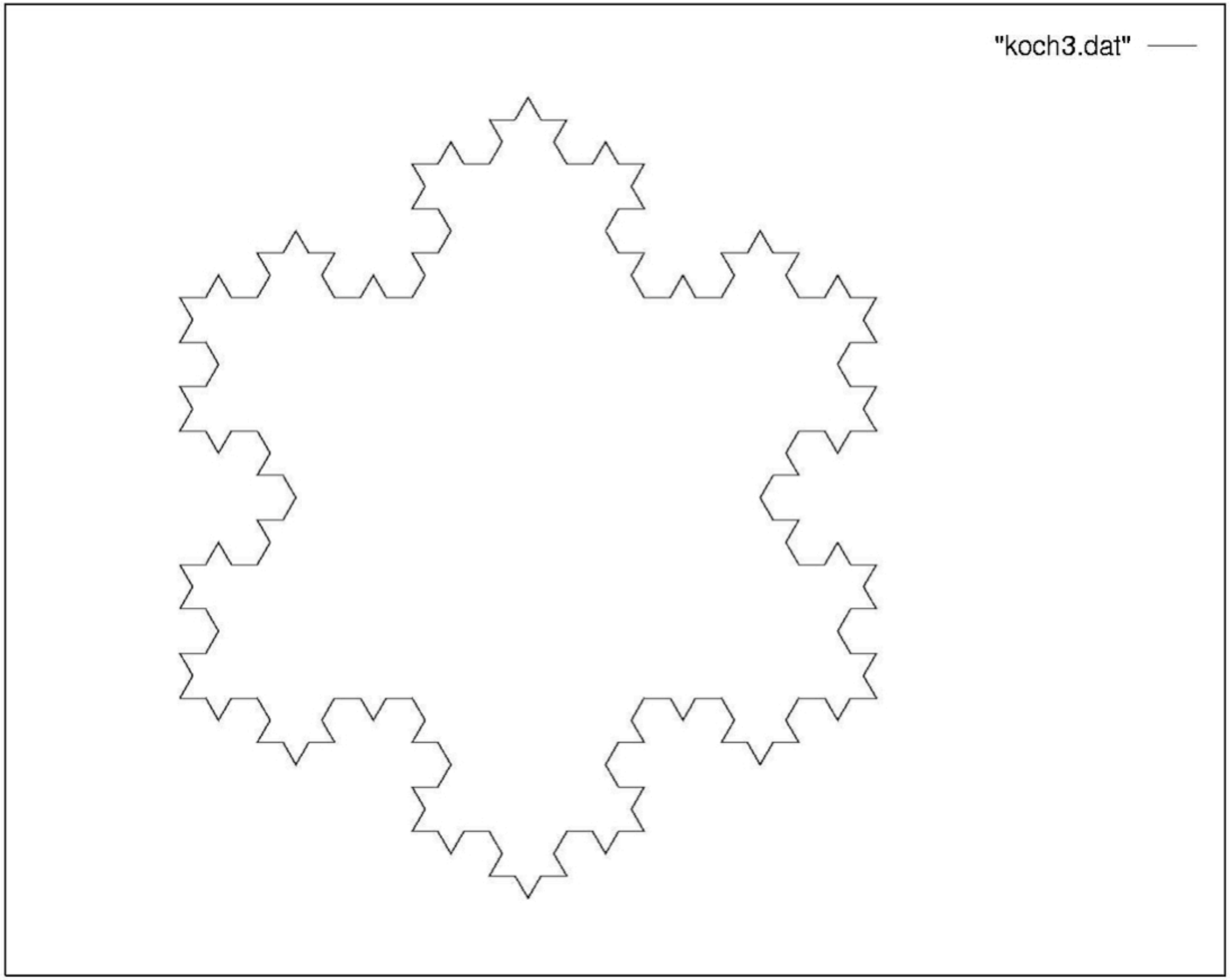
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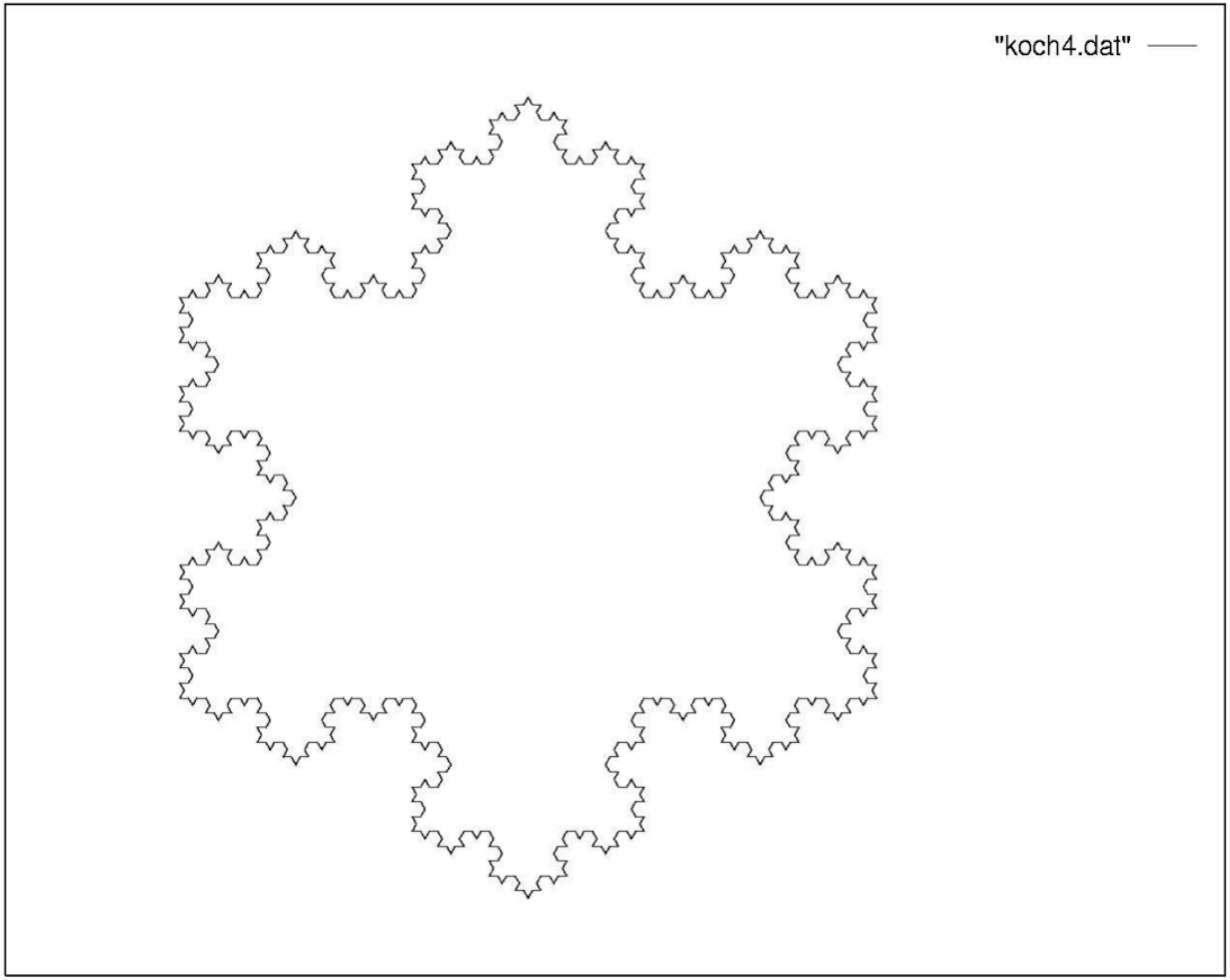
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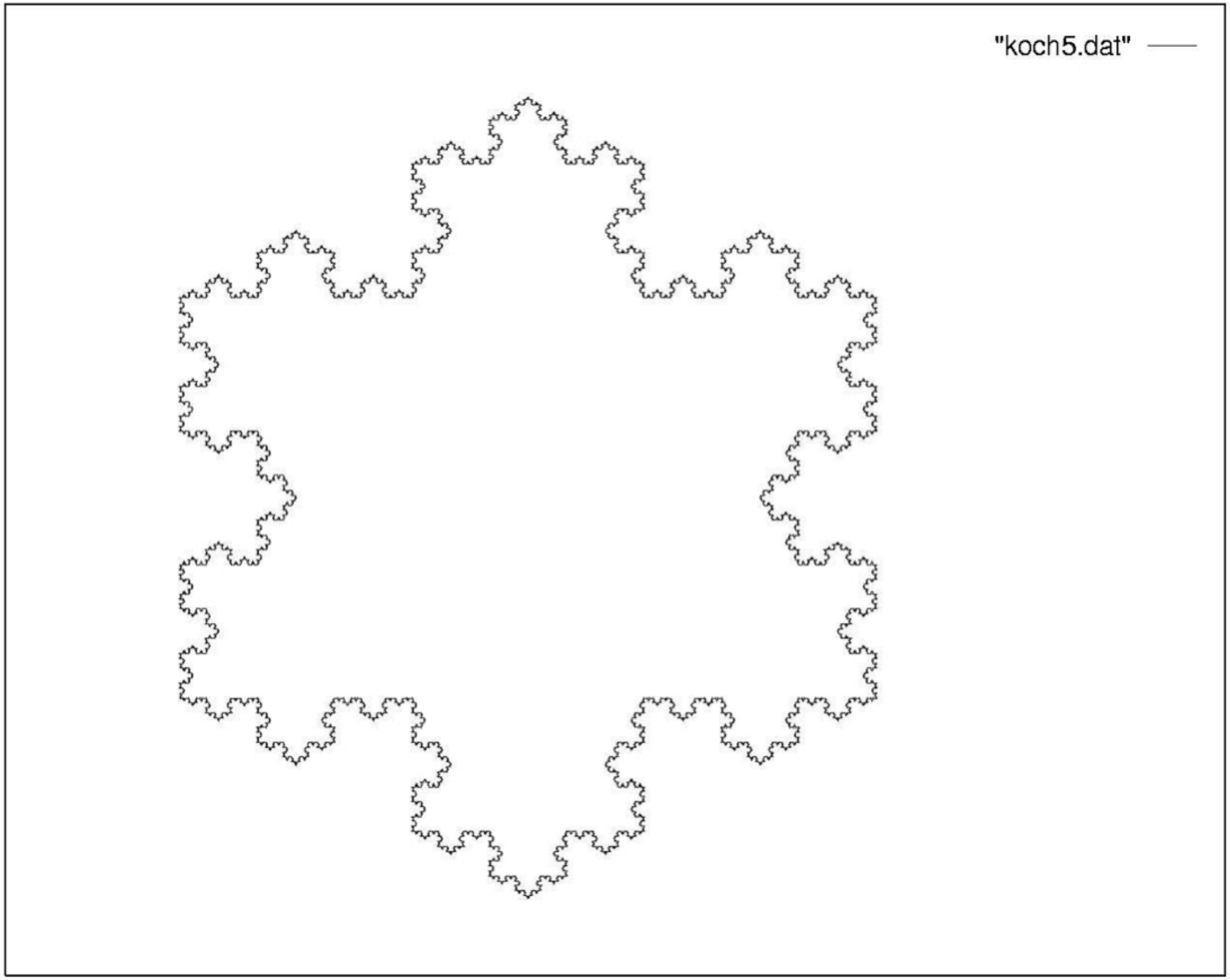
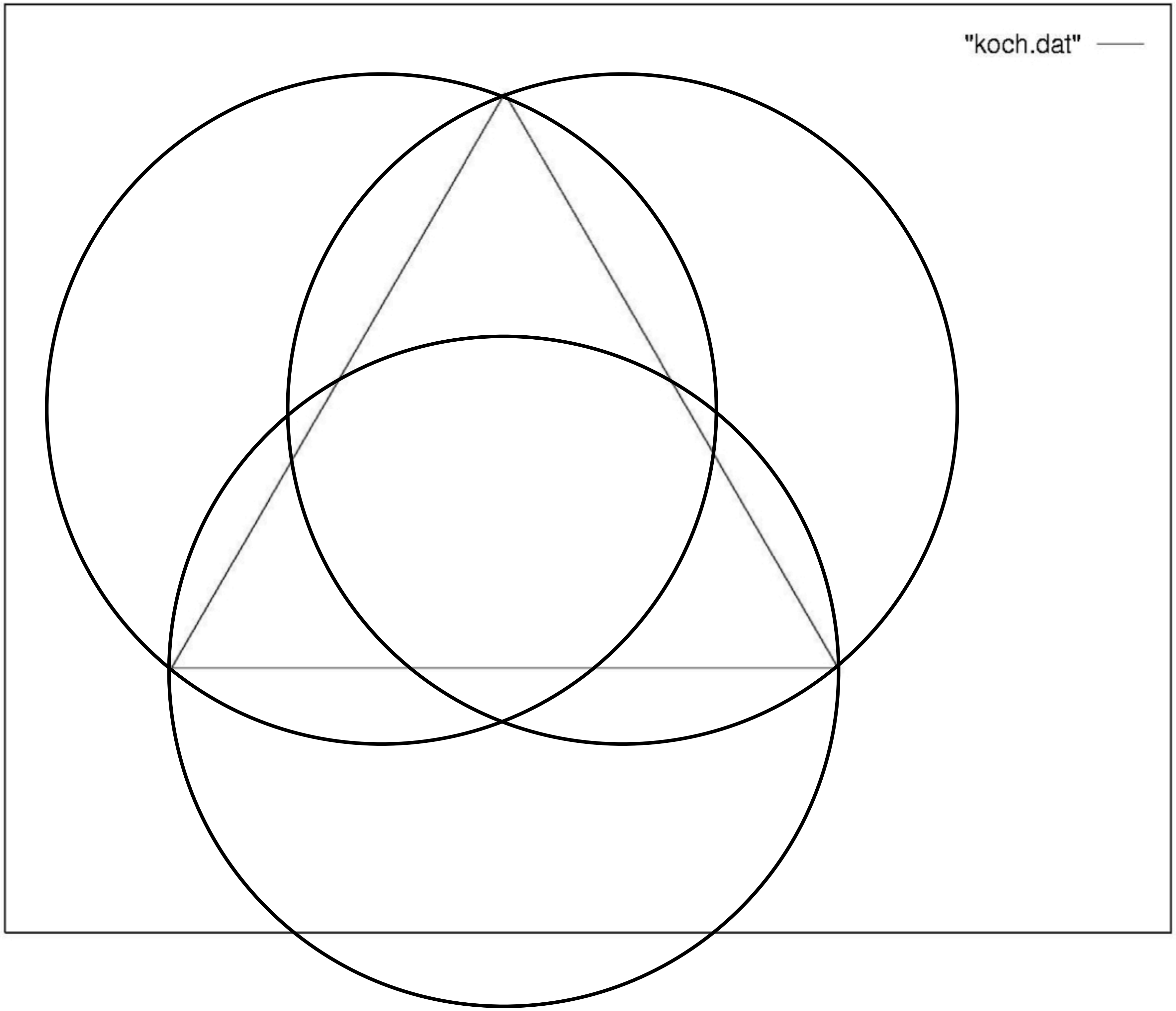

"koch5.dat"

- To calculate the fractal / Hausdorff / capacity / box-counting dimension, we again see how many boxes (circles) of diameter (again)1/3^n we need to cover the set.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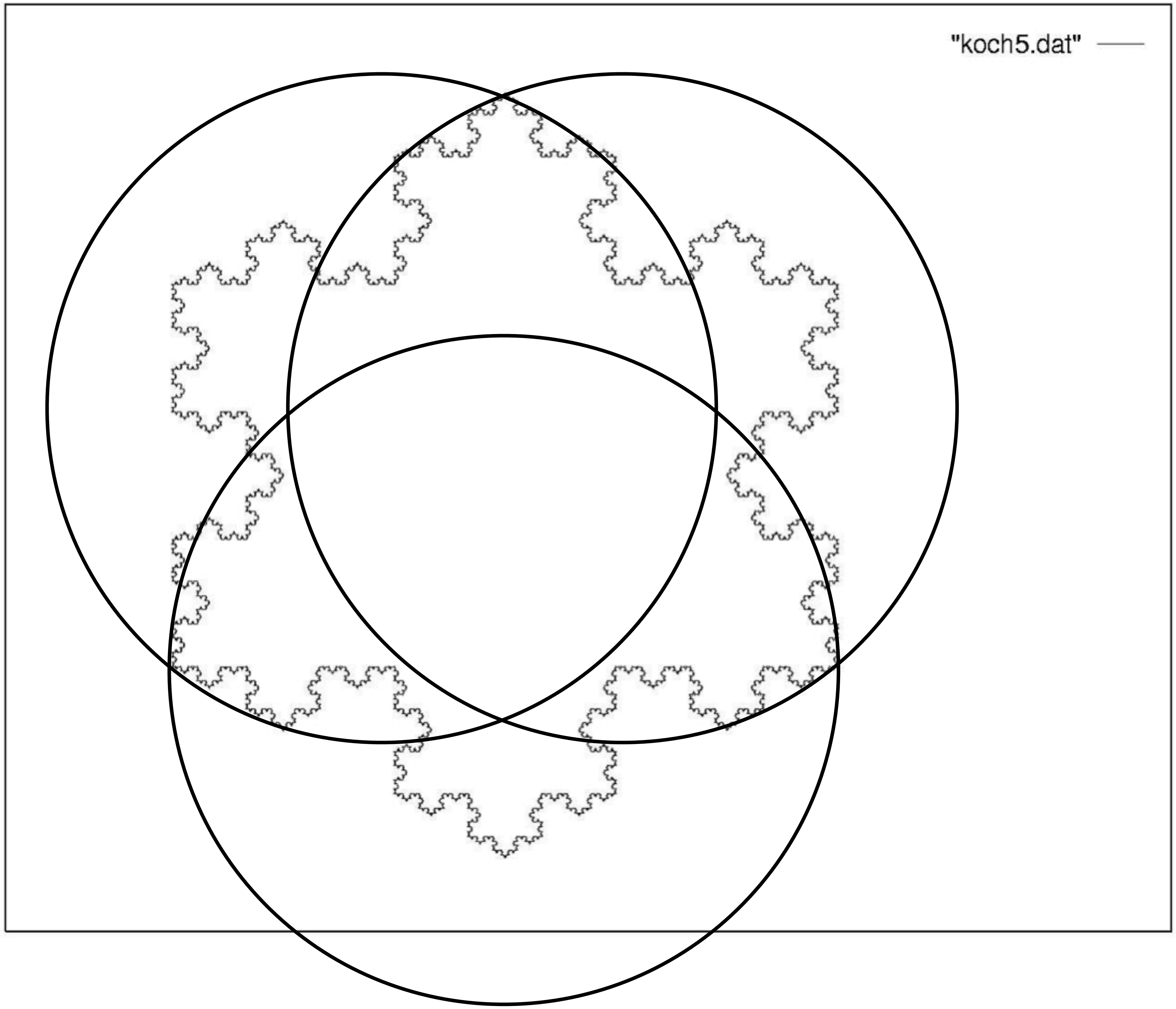
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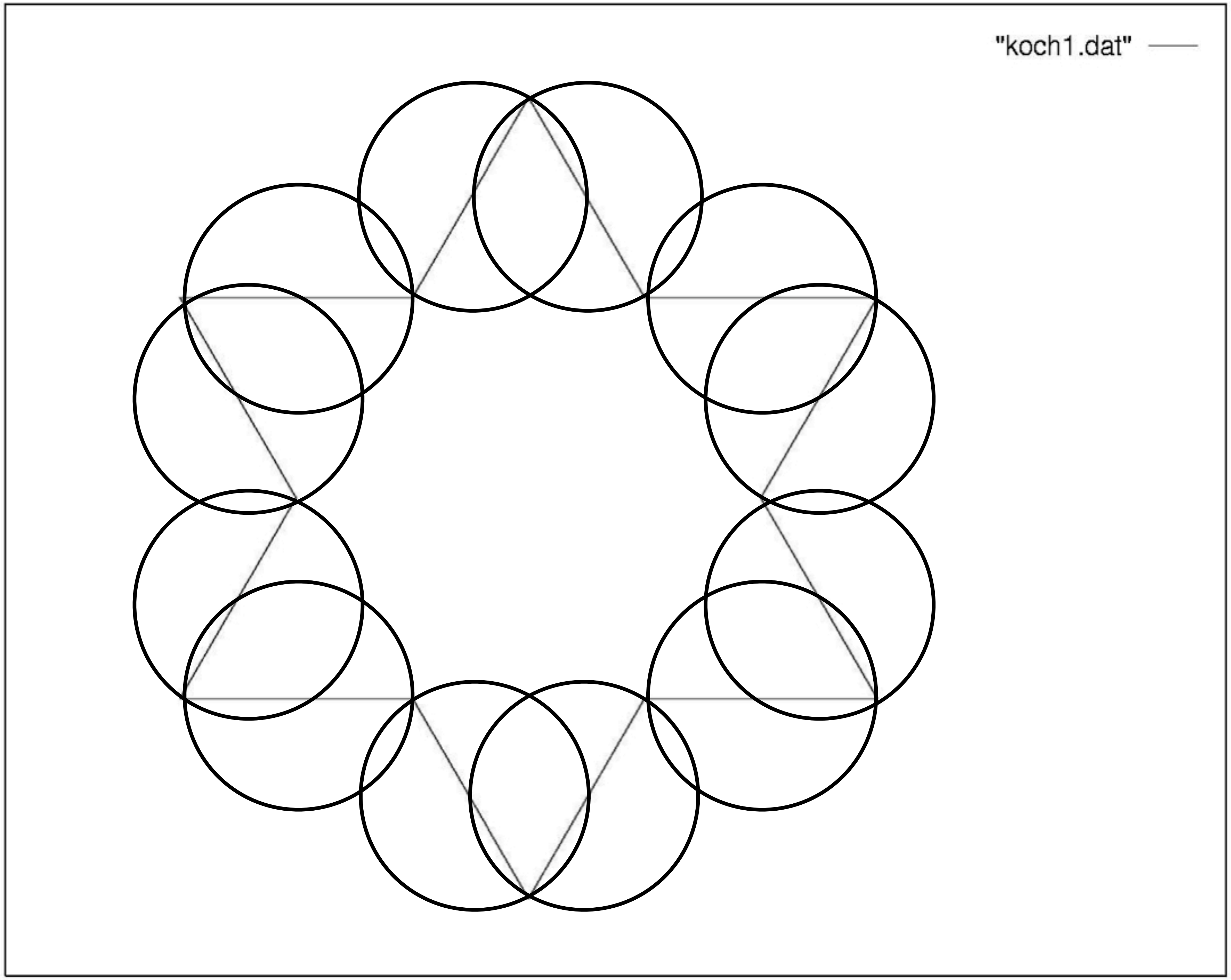
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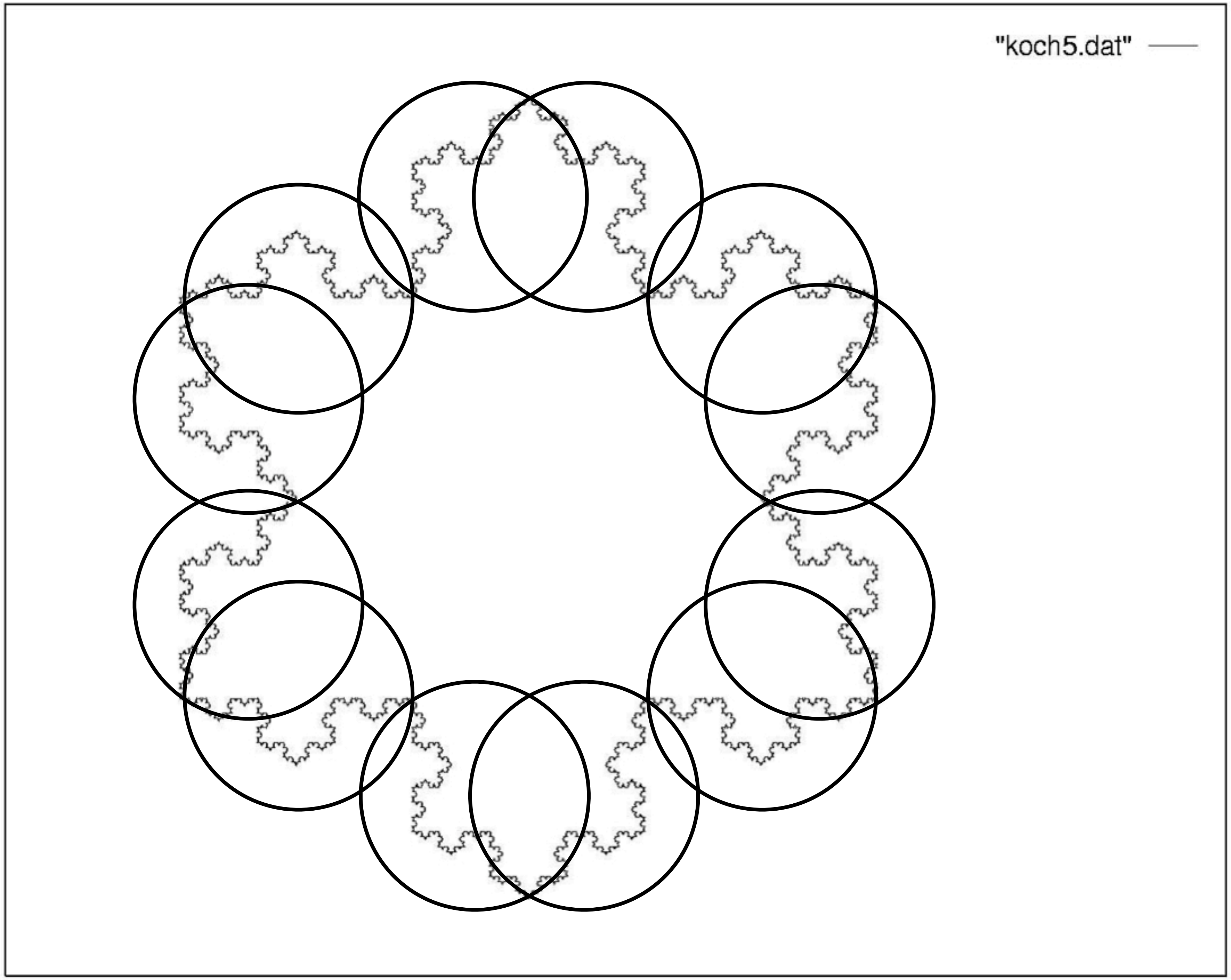
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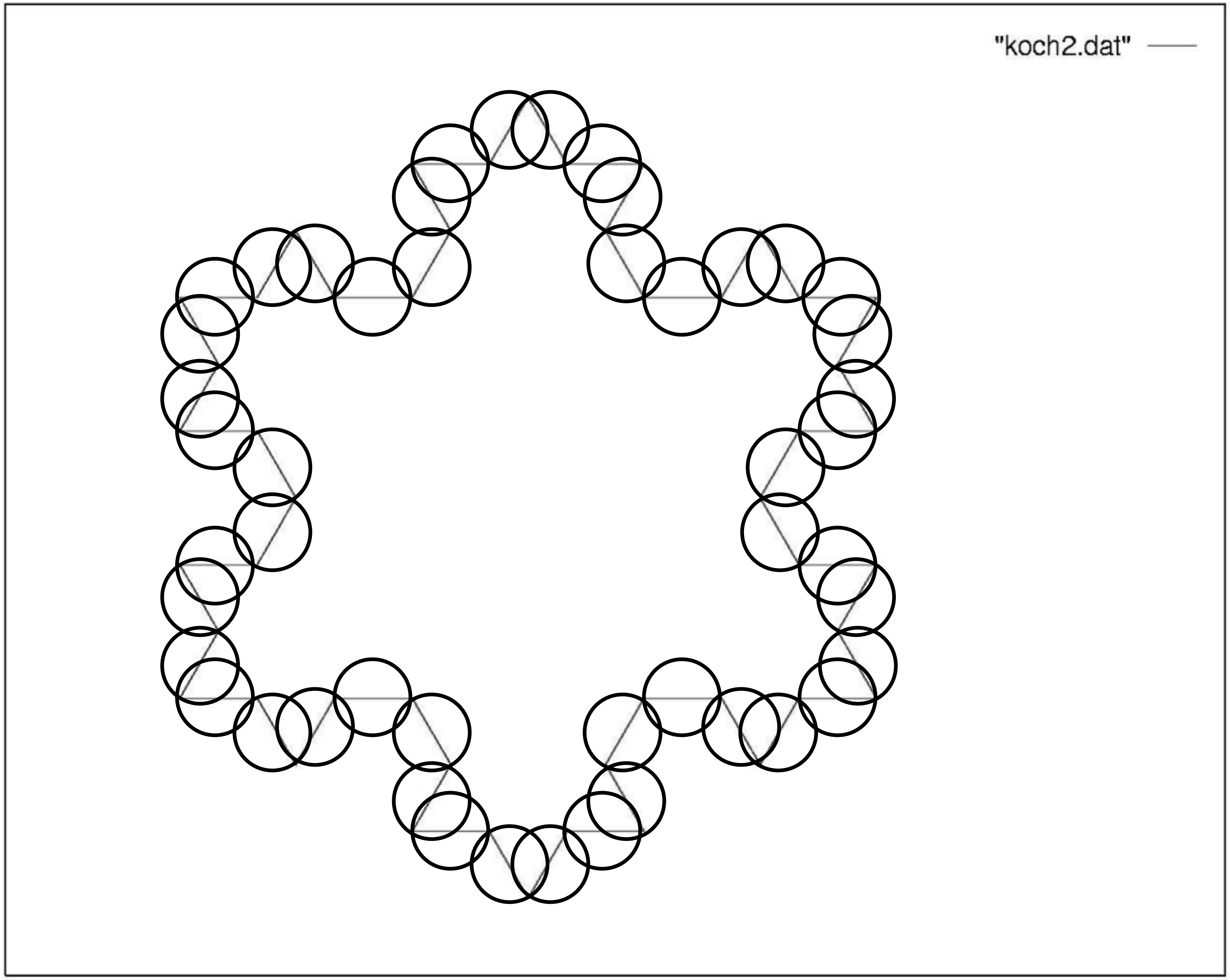
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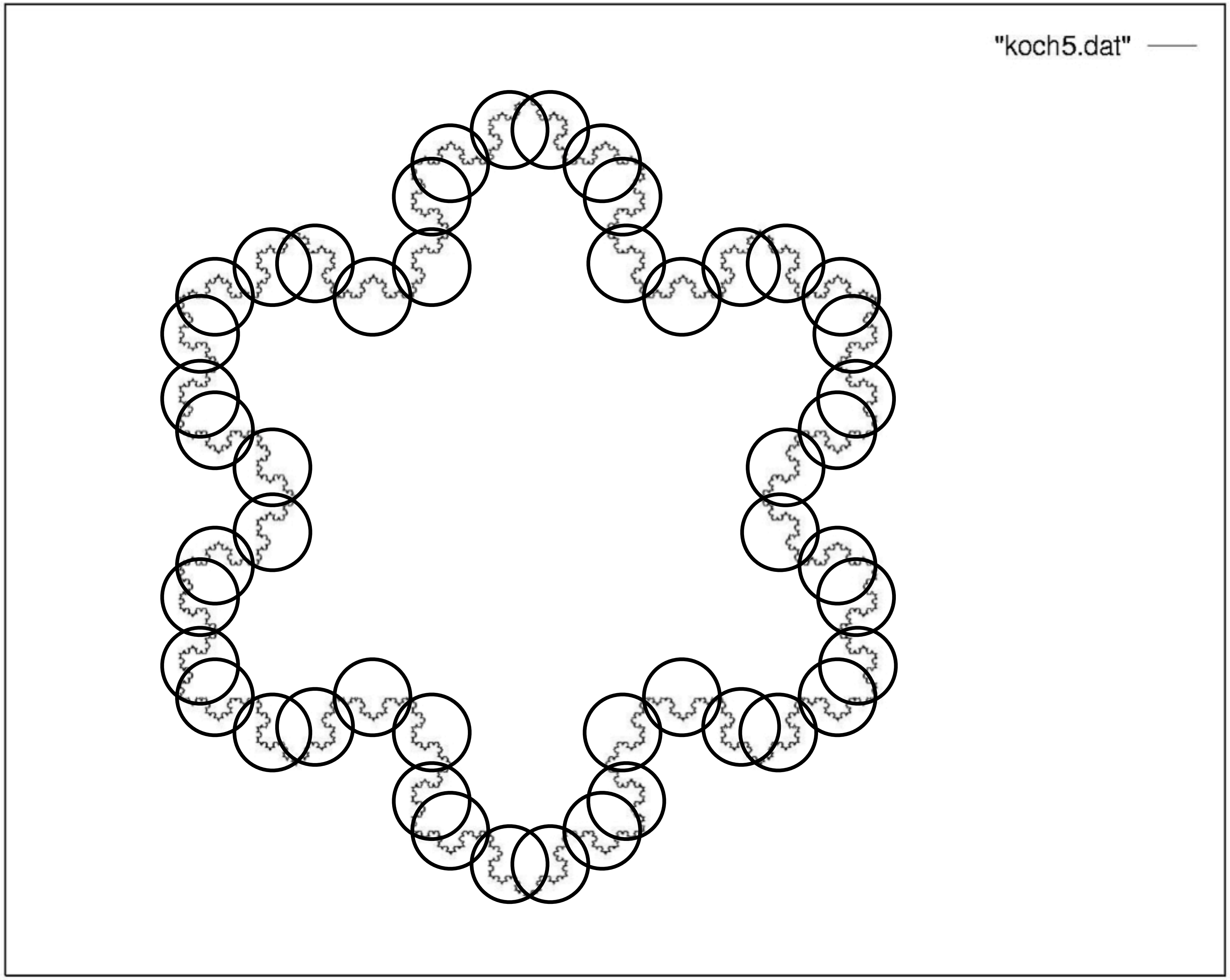
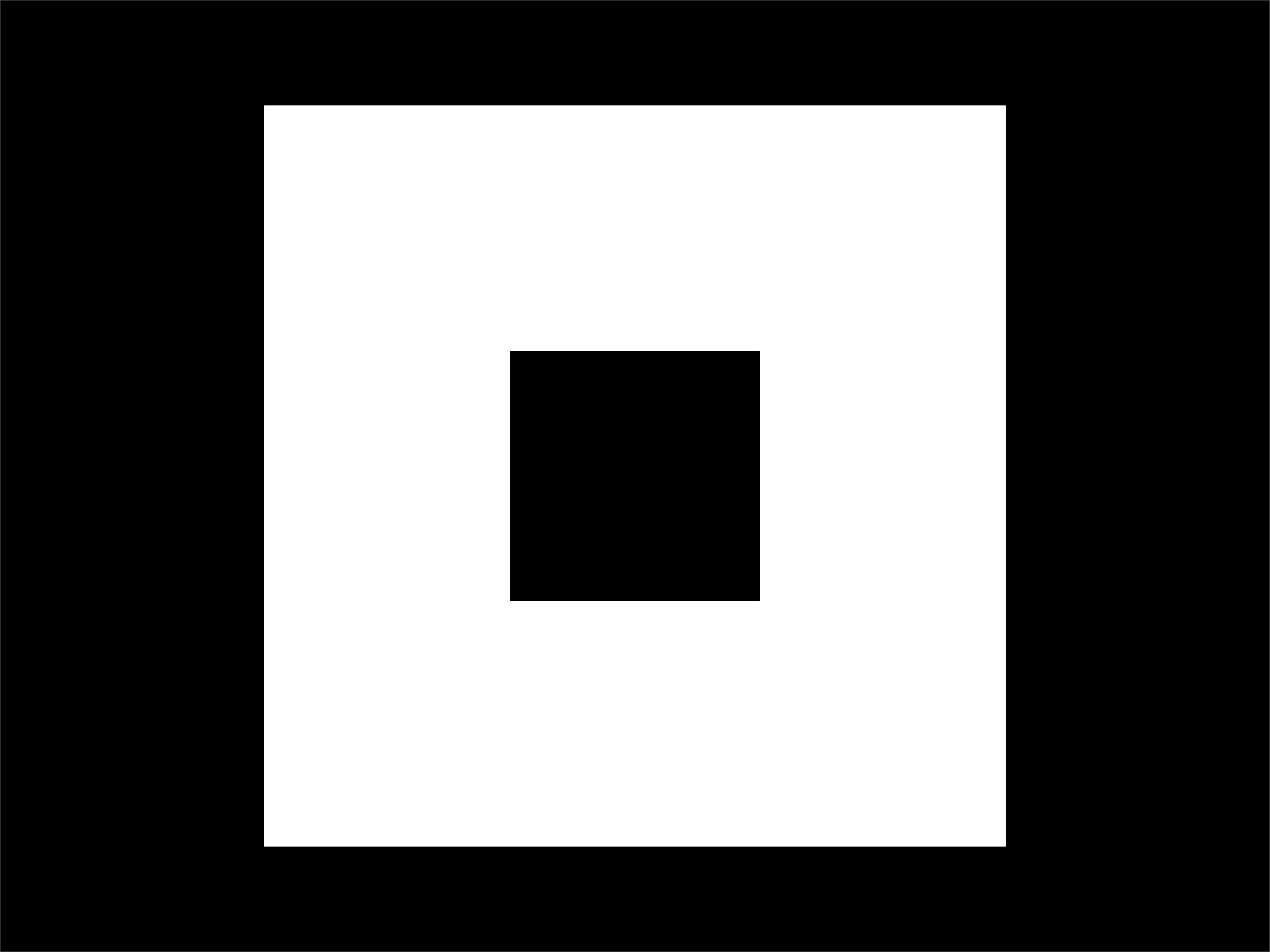
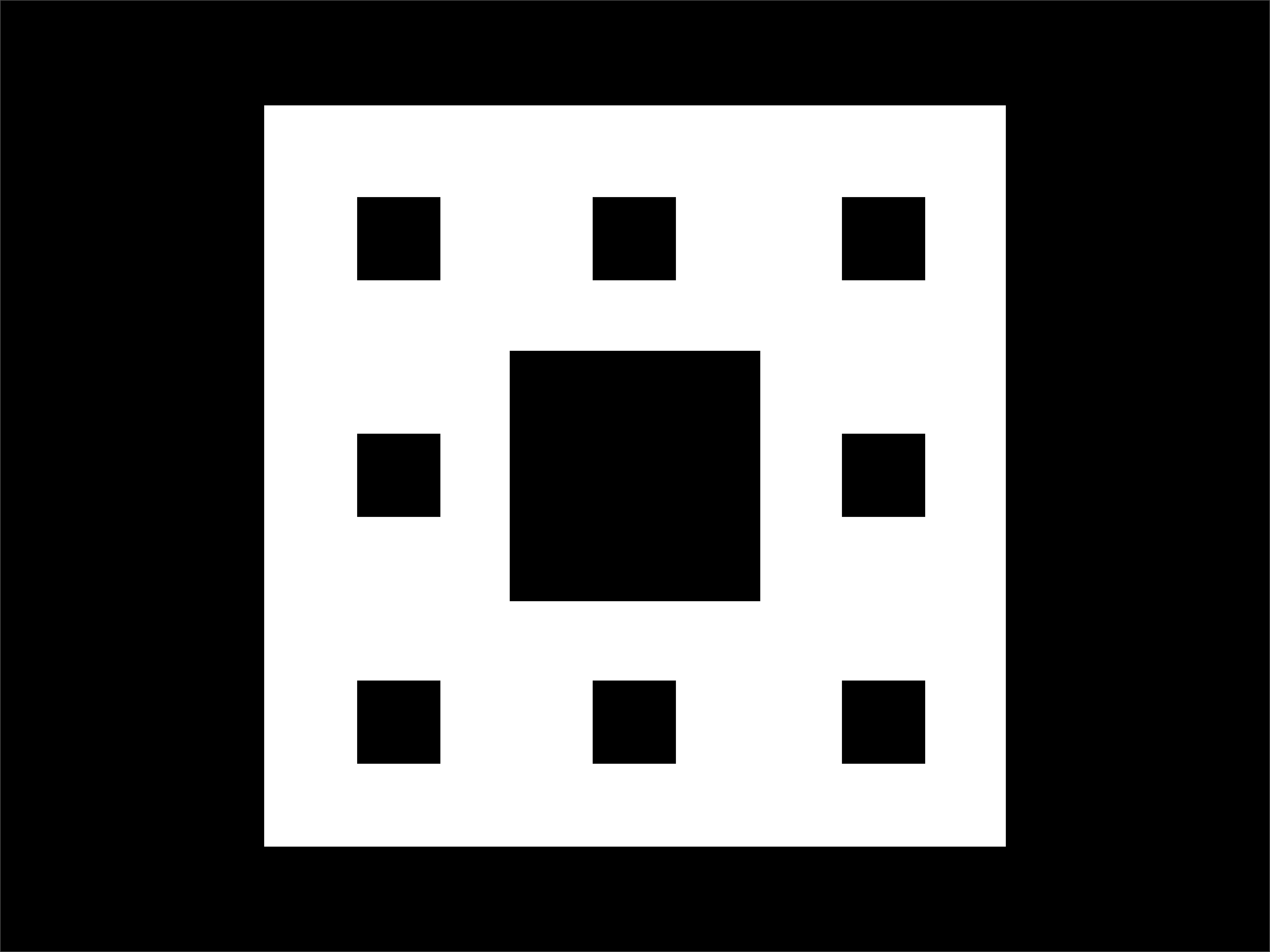

"koch5.dat"

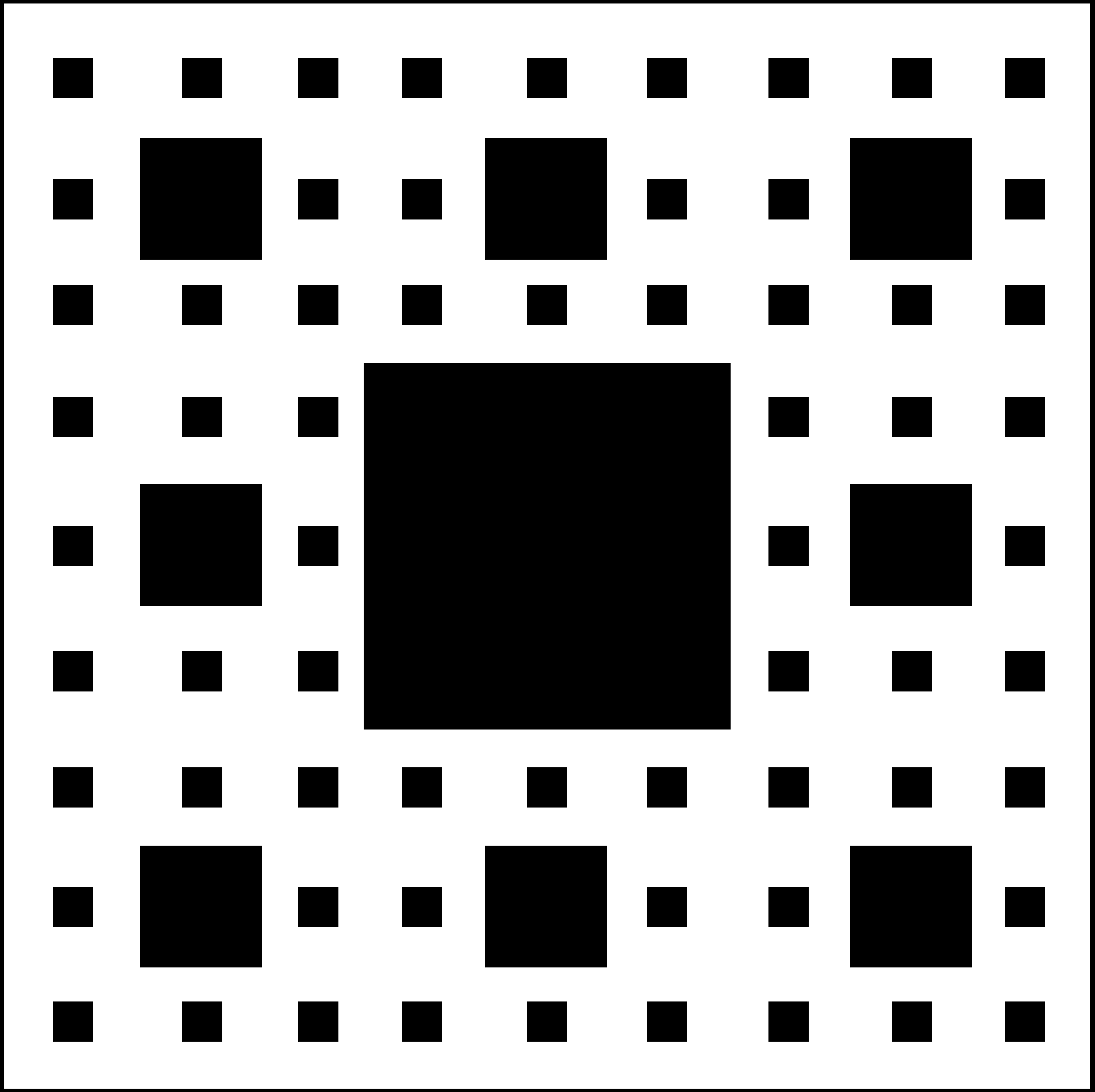
$$D = \mathrm{Lim}(\log(N_r)/\log(1/r))$$
$$= \log(4) / \log(3)$$

r	N_r
1	3
1/3	3 * 4
1/3^2	3 * 4^2
1/3^3	3 * 4^3
1/3^n	3 * 4^n

- The Sierpinski carpet: We start with a square. We remove the middle square with side one third. For each of the remaining squares of side one third, remove the central square. We repeat the process.



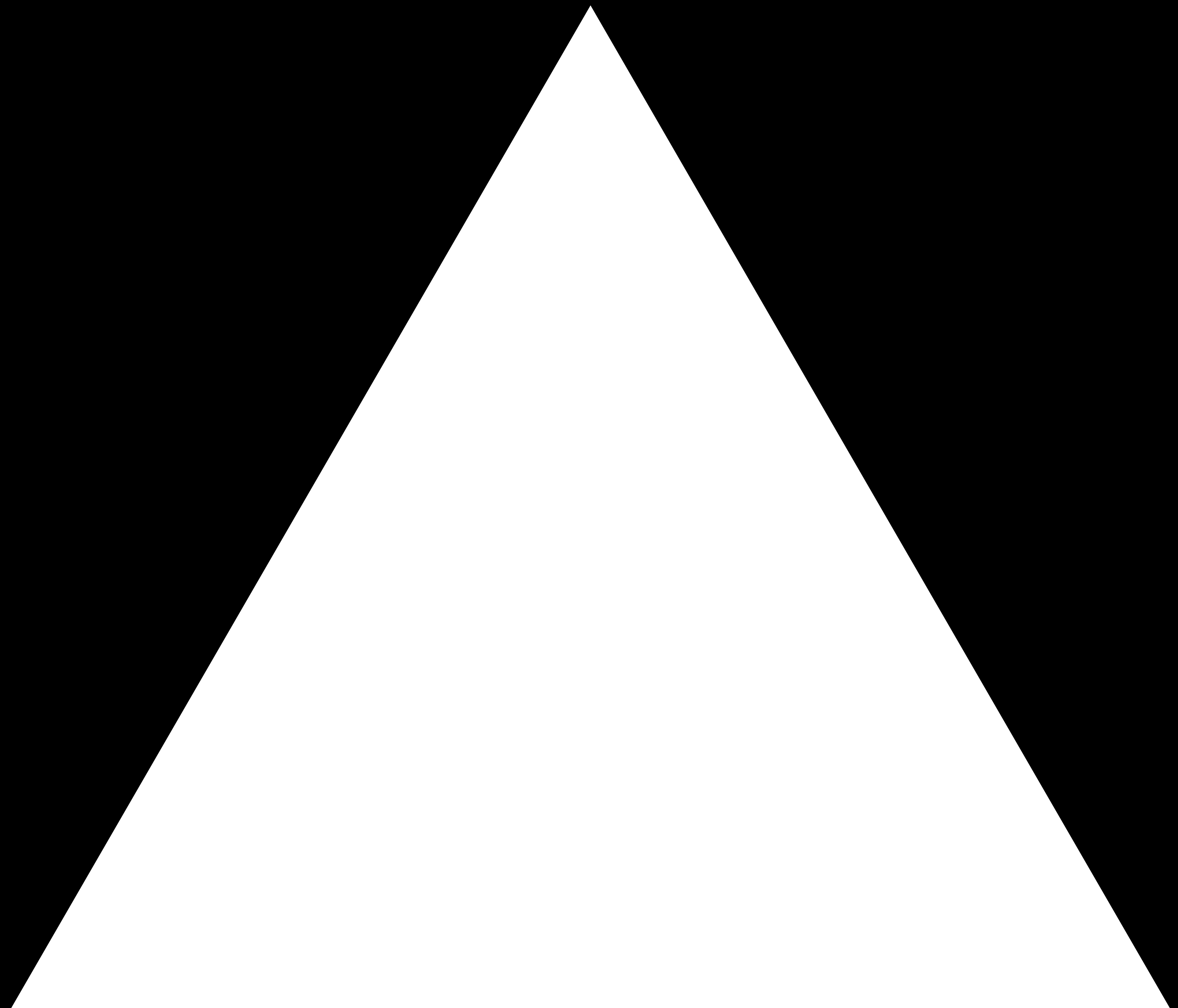


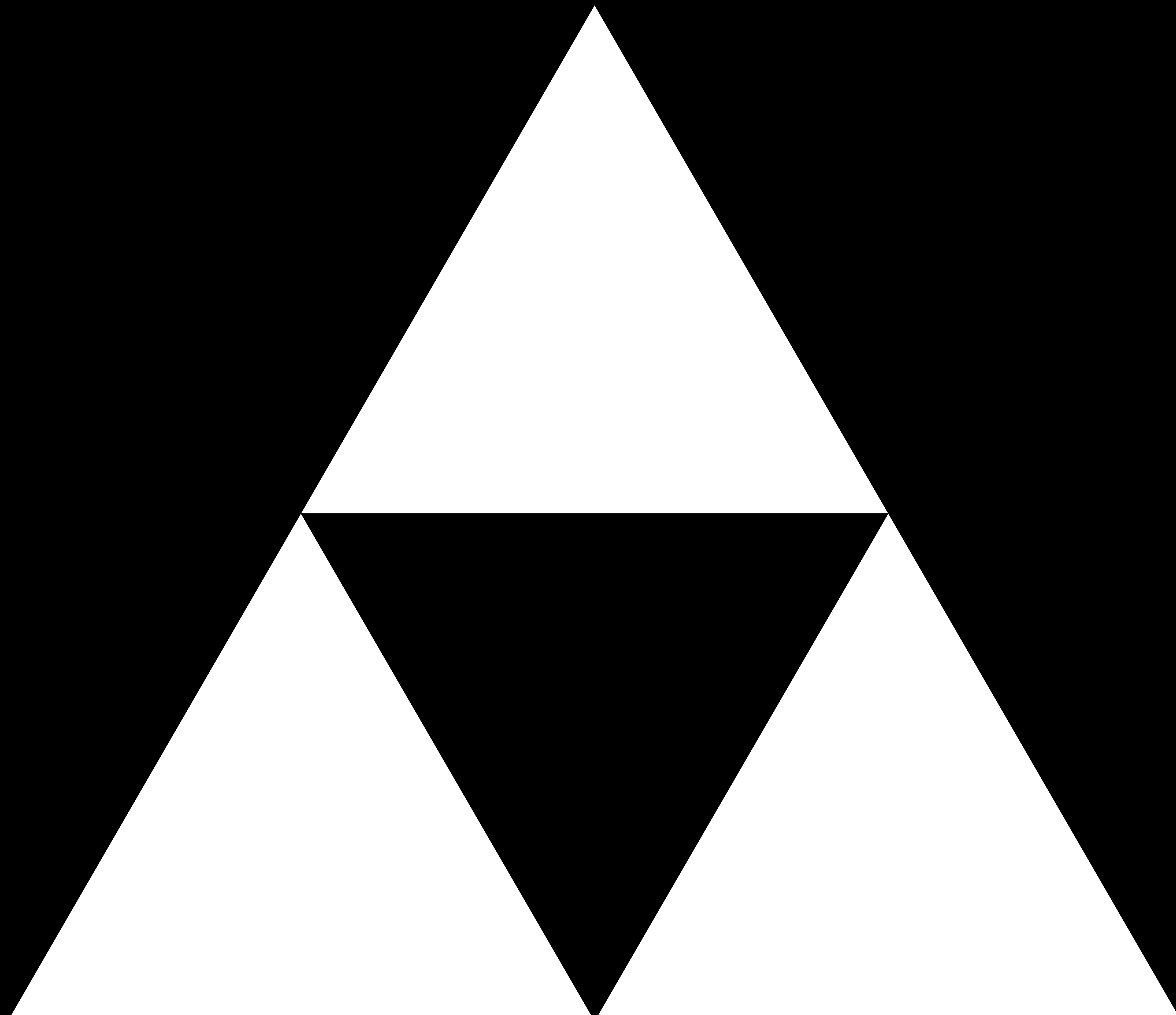


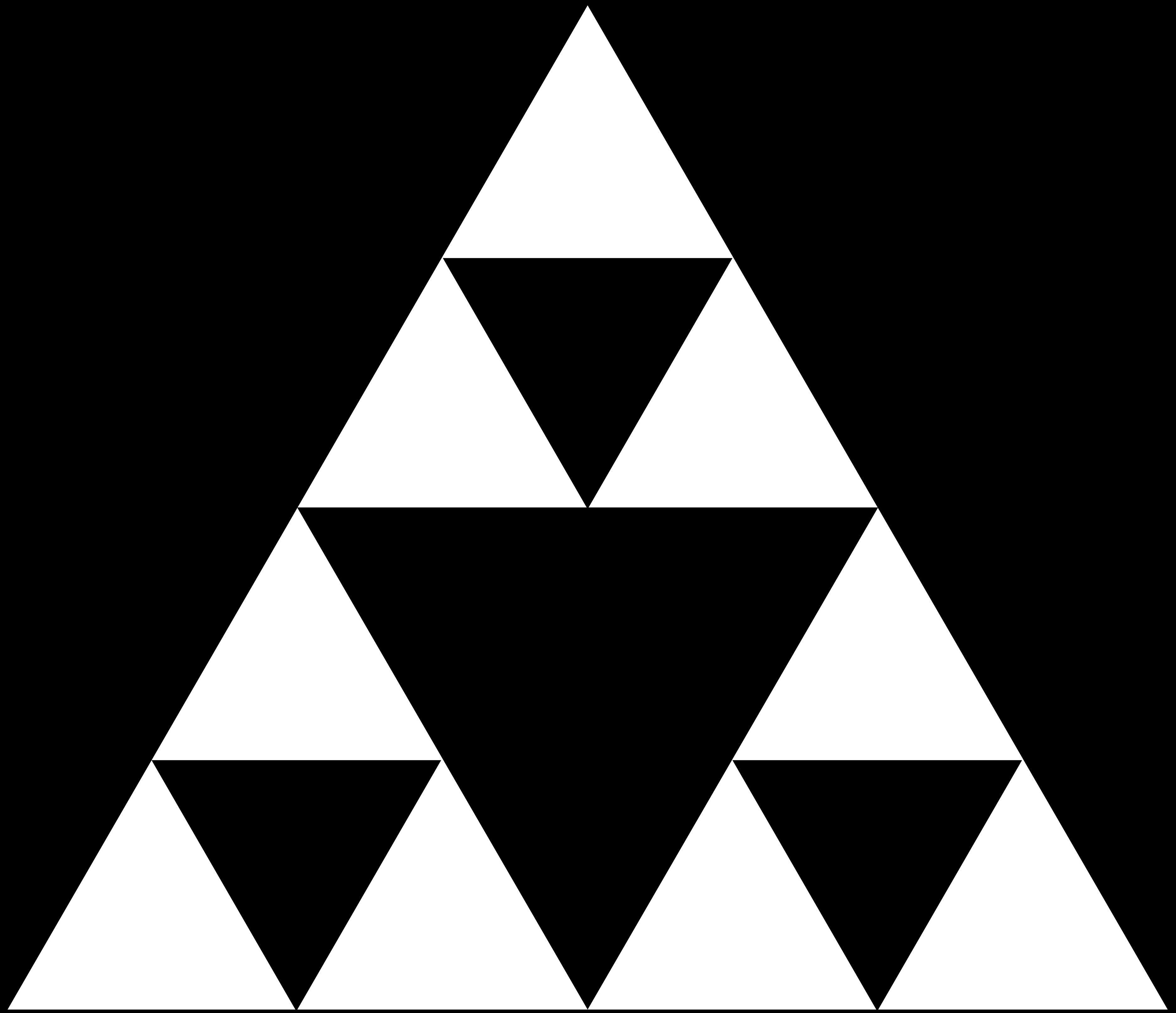
$$D = \mathrm{Lim}(\log(N_r)/\log(1/r))$$
$$= \log(8) / \log(3)$$

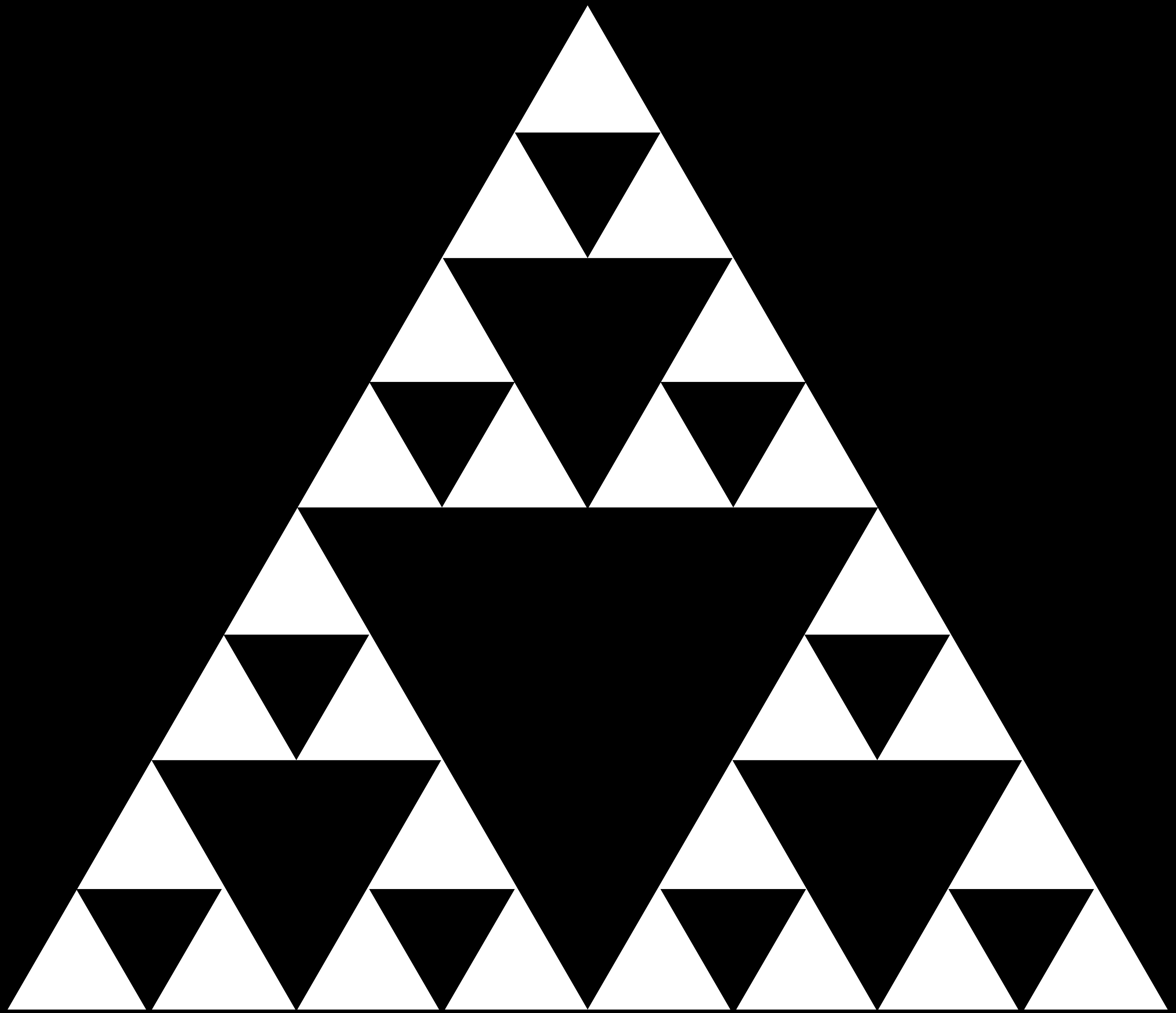
r	N_r
1	1
1/3	8
1/3^2	8^2
1/3^3	8^3
1/3^n	8^n

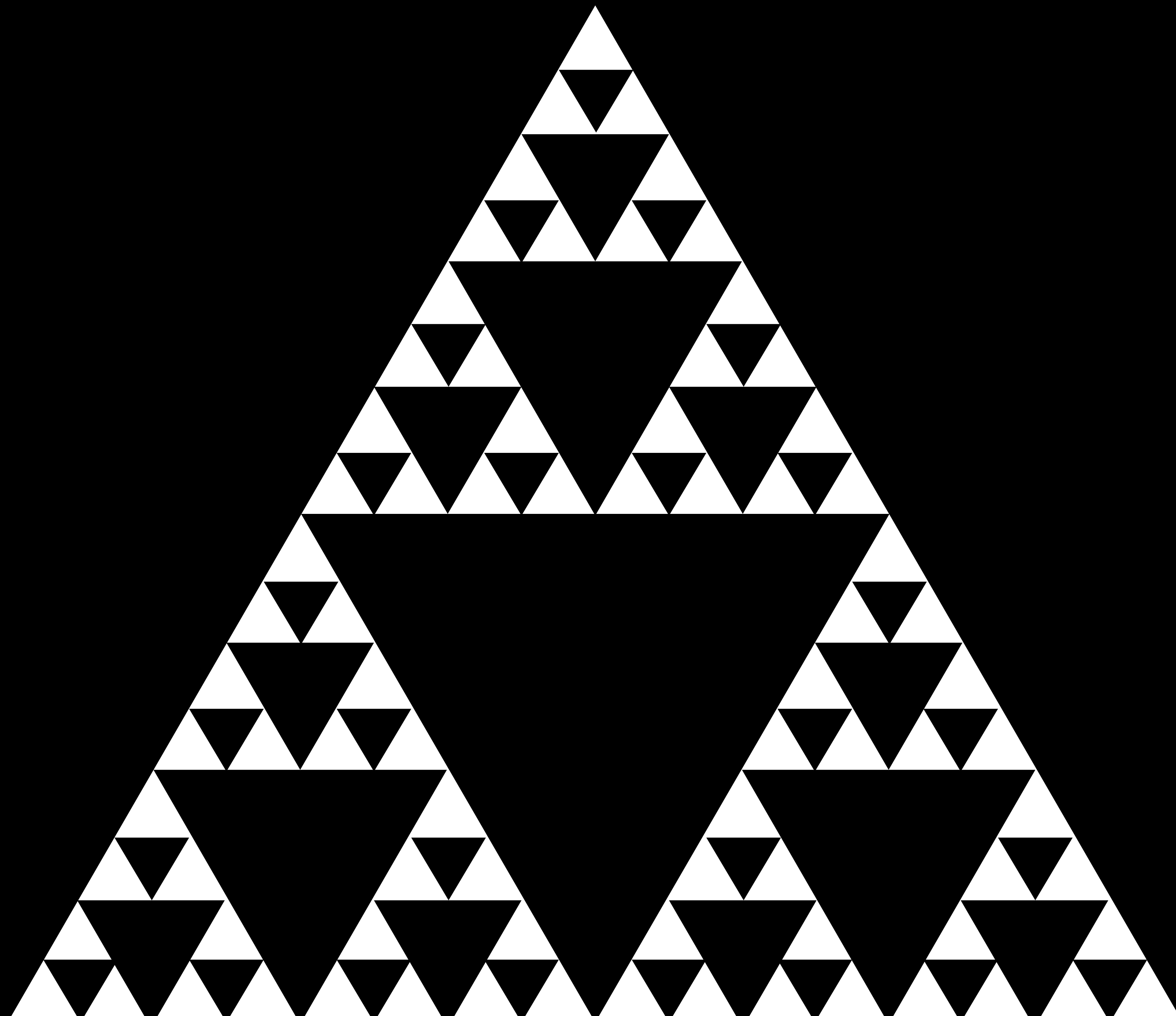
- The Sierpinski gasket: we do a similar process with an equilateral triangle, removing a central triangle. (Note: we could also do a similar thing taking cubes out of a larger cube -- the Sierpinski sponge -- but it's hard to draw :-)







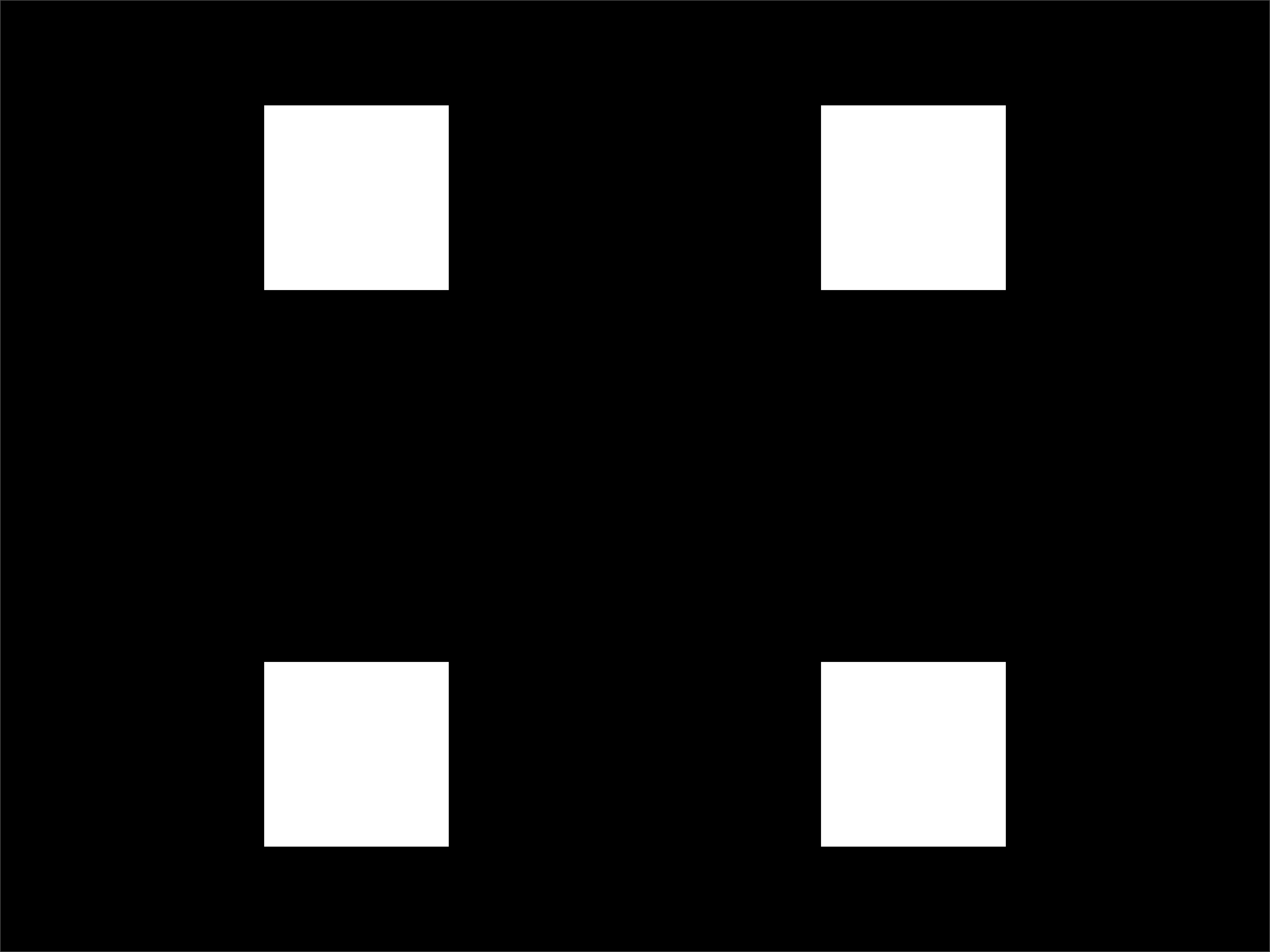


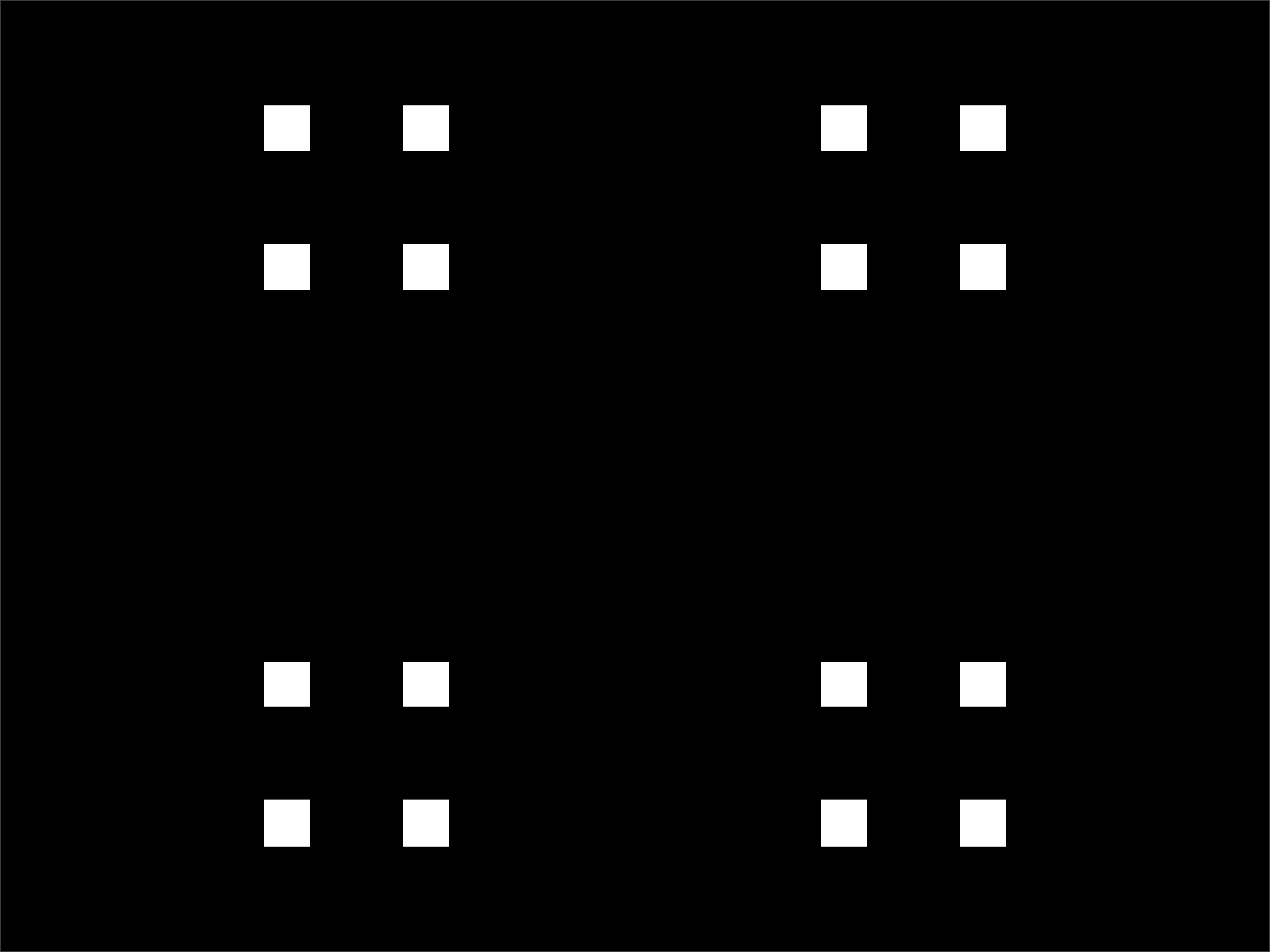


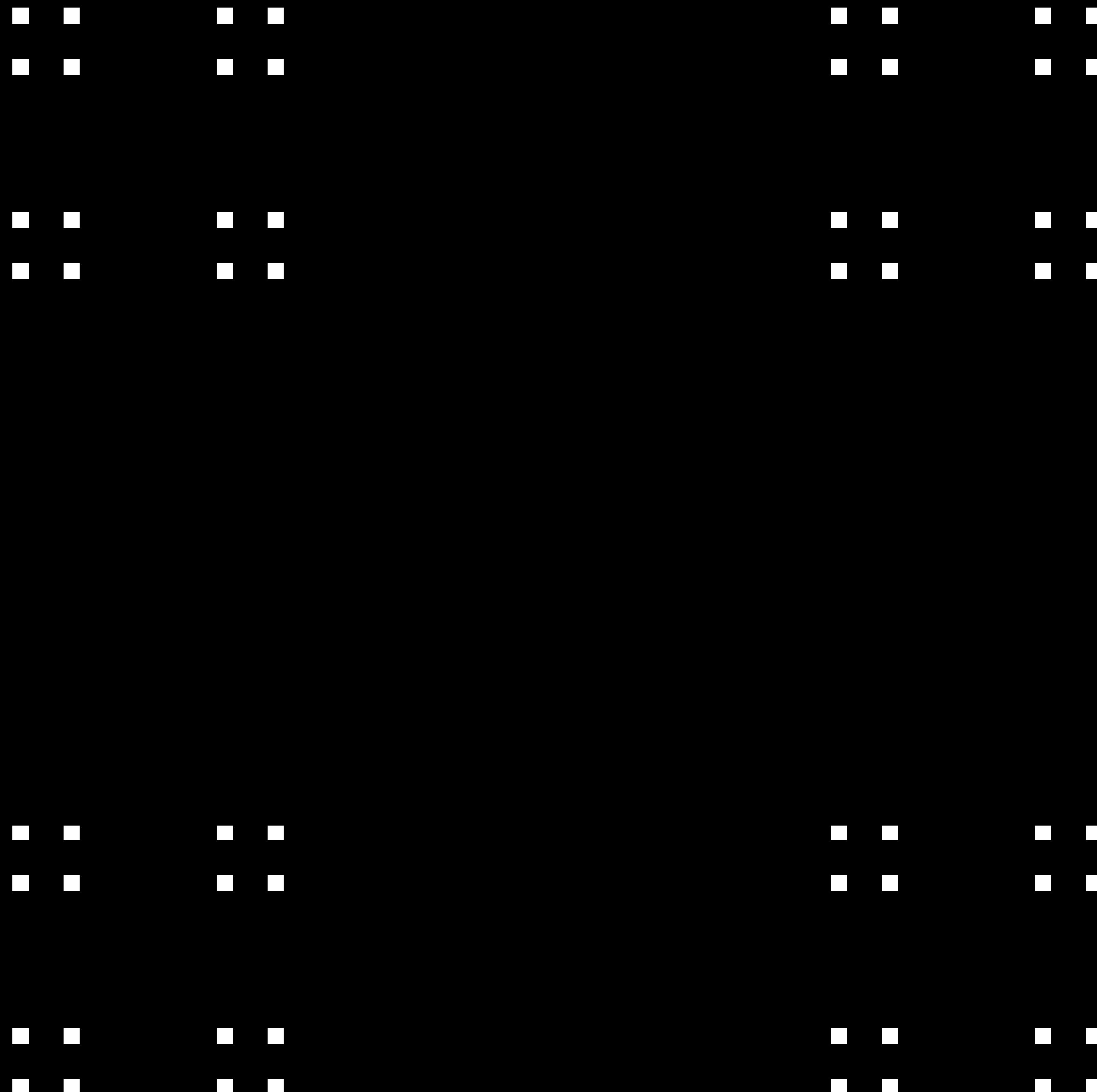
$$D = \mathrm{Lim}(\log(N_r)/\log(1/r))$$
$$= \log(3) / \log(2)$$

r	N_r
1	1
1/2	3
1/2^2	3^2
1/2^3	3^3
1/2^n	3^n

- We can also remove other shapes.

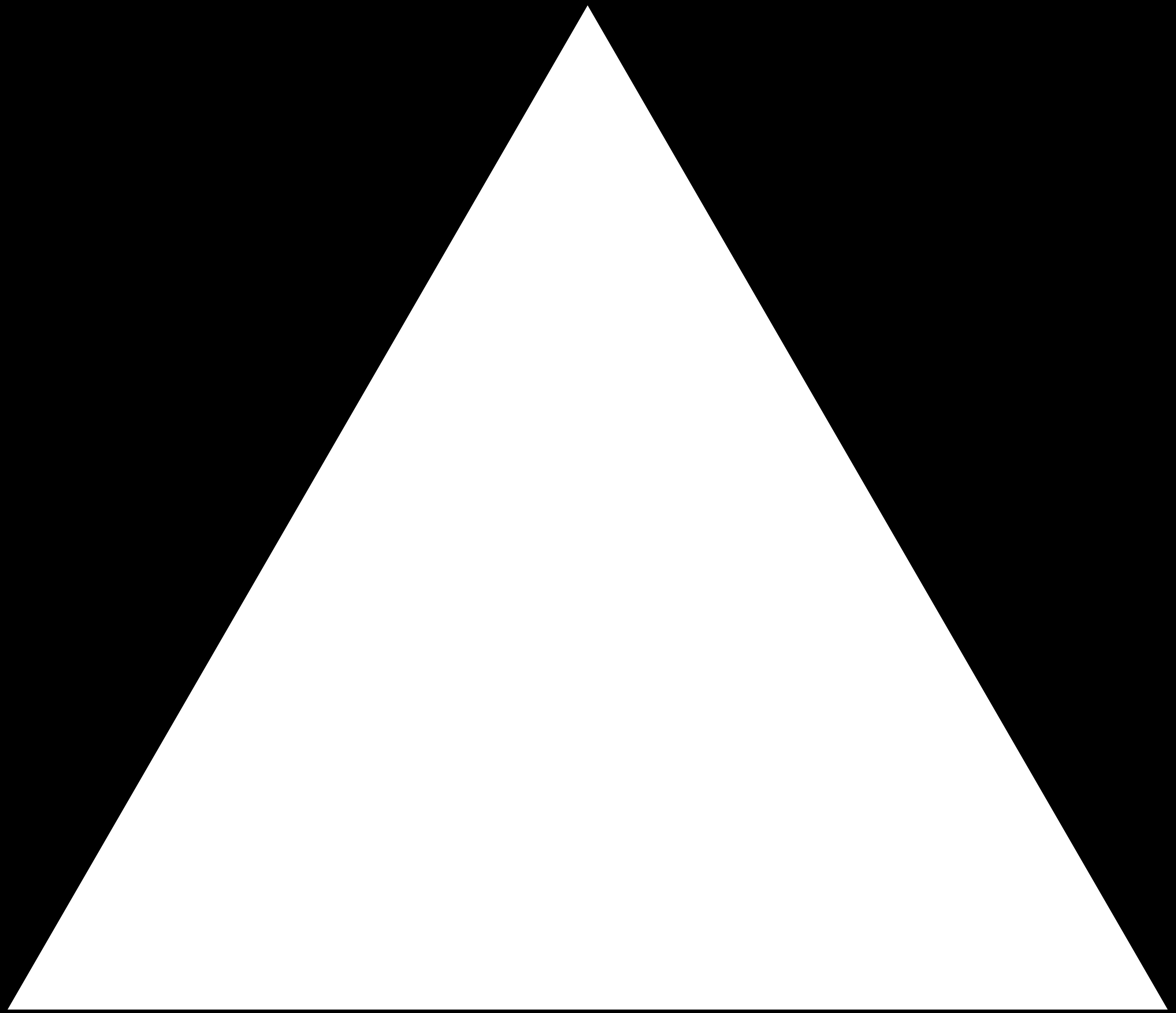


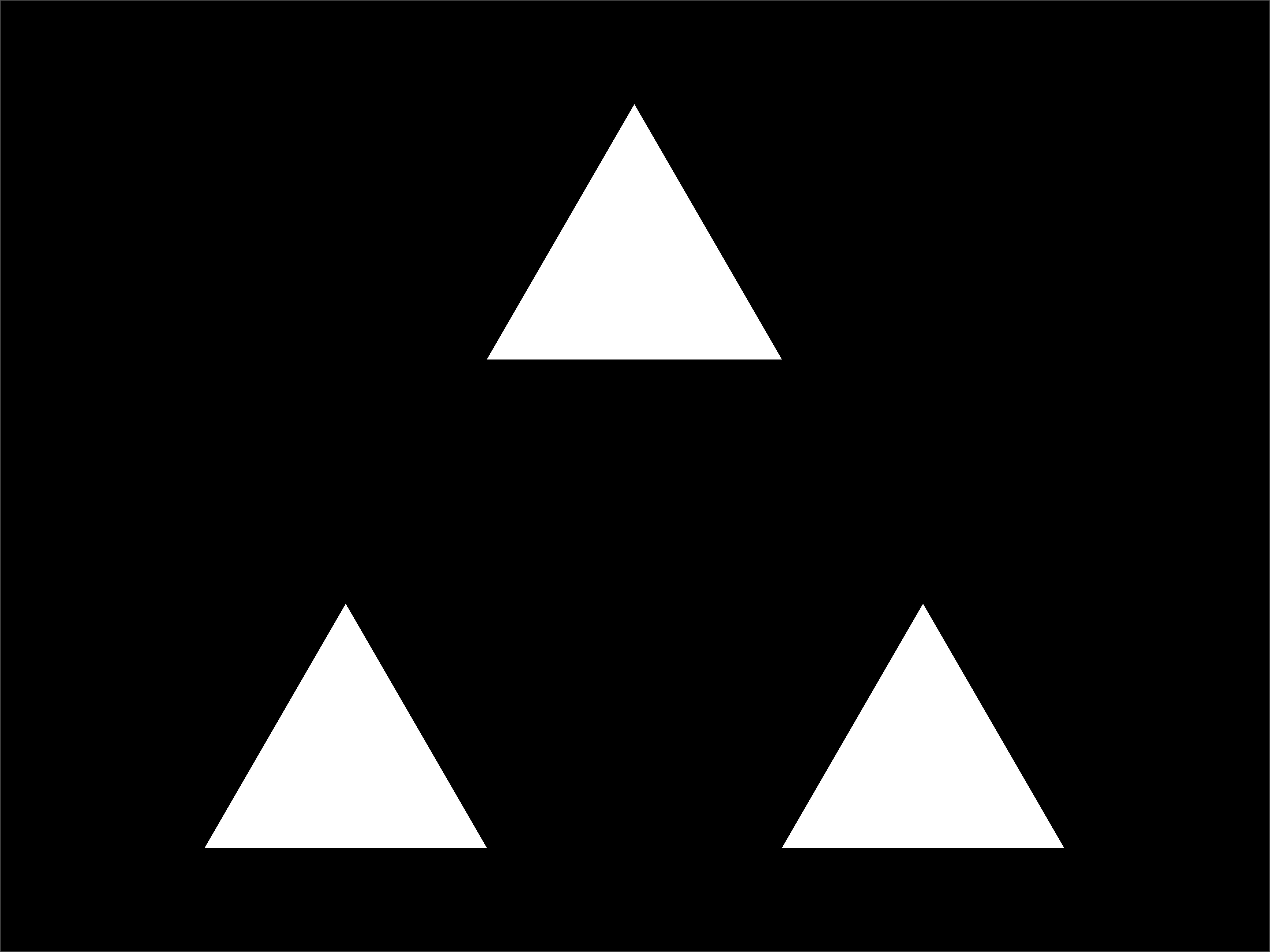


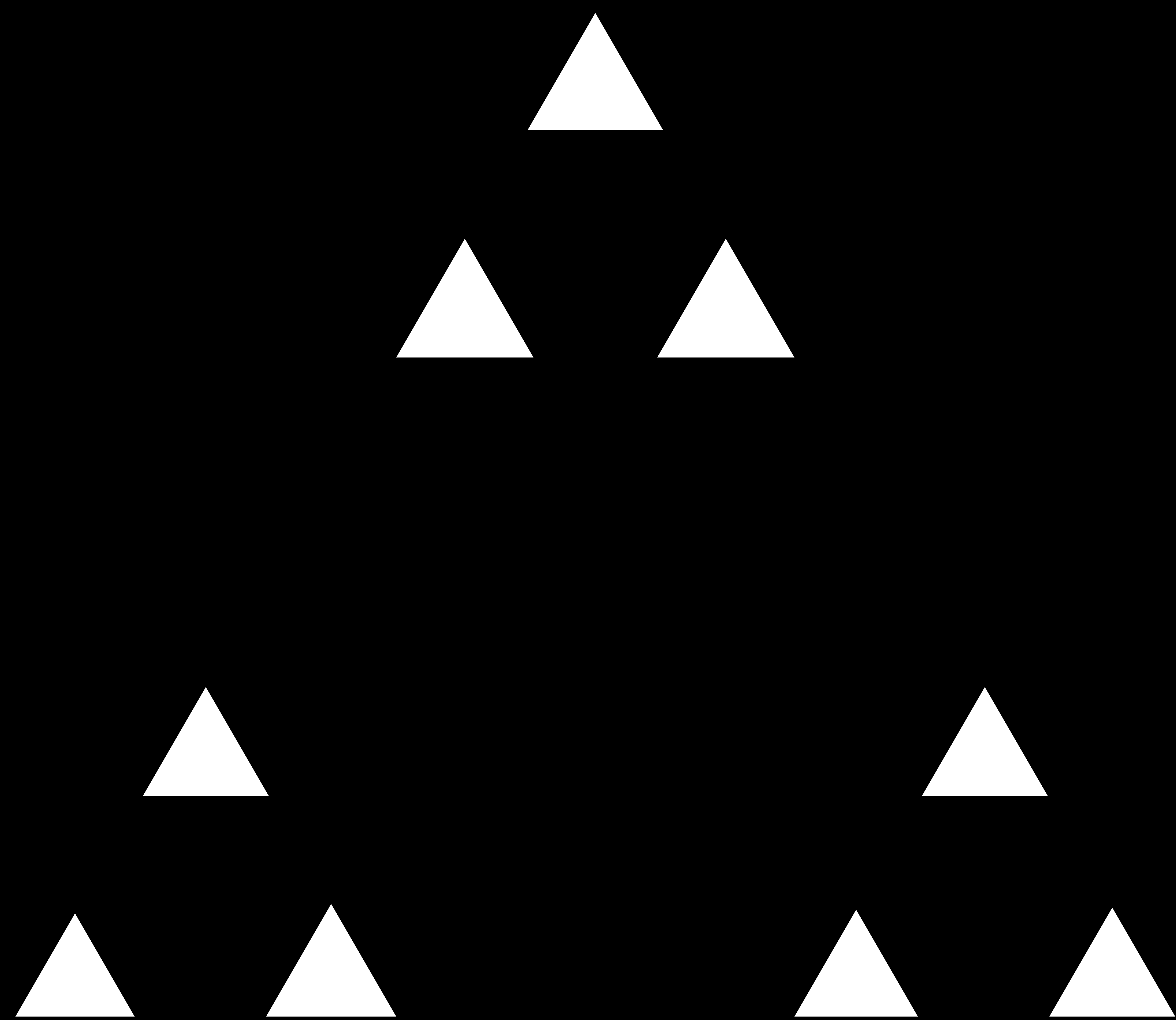


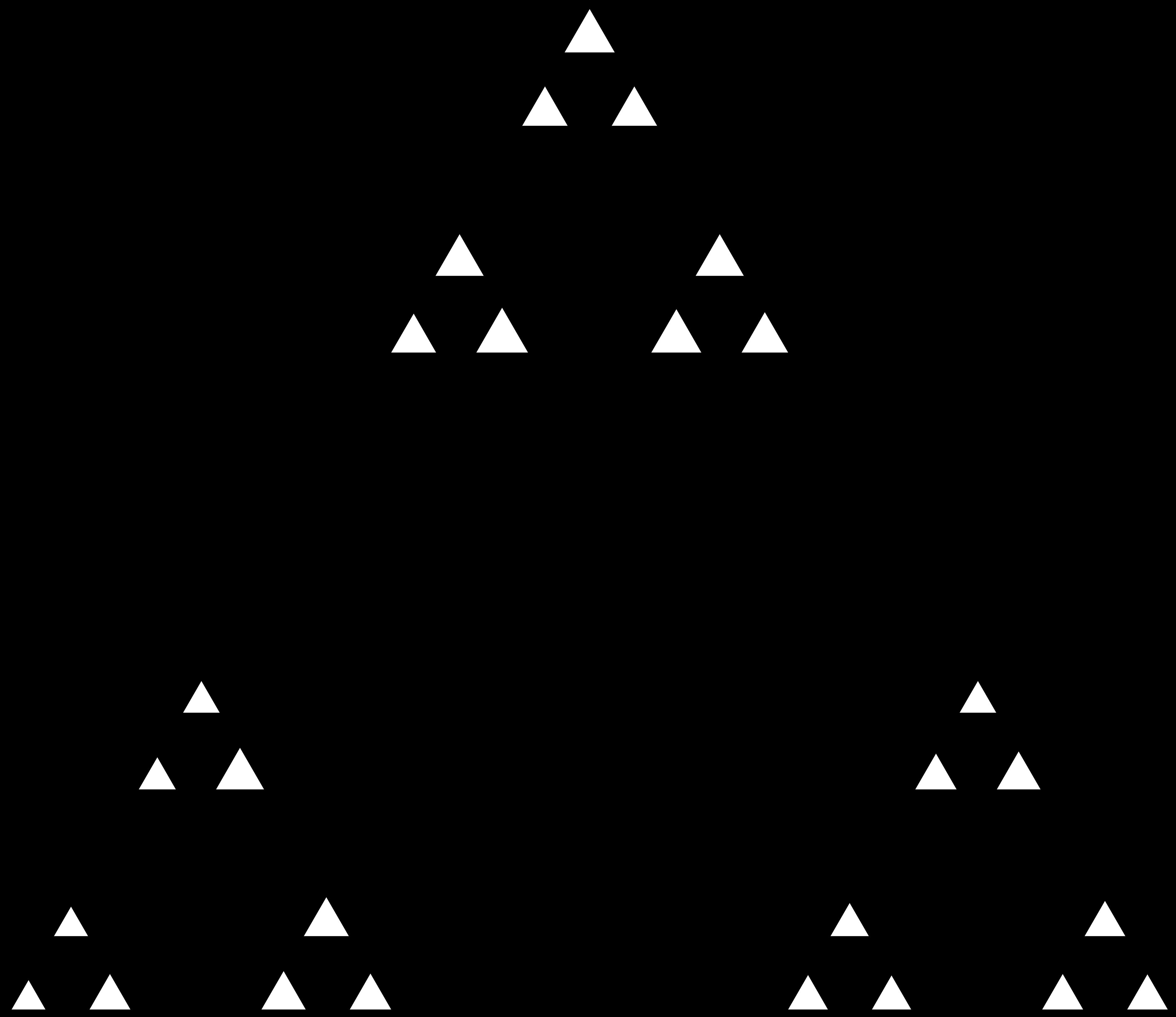
$$D = \mathrm{Lim}(\log(N_r)/\log(1/r))$$
$$= \log(4) / \log(4) = 1$$

r	N_r
1	1
1/4	4
1/4^2	4^2
1/4^3	4^3
1/4^n	4^n









$$D = \mathrm{Lim}(\log(N_r)/\log(1/r))$$
$$= \log(3) / \log(3) = 1$$

r	N_r
1	1
1/3	3
1/3^2	3^2
1/3^3	3^3
1/3^n	3^n

Relating this to nonlinear dynamical systems

Suppose we have a dissipative dynamical system (continuous) with positive Lyapunov exponent -- for ease of viewing, let's suppose it is 2-dimensional . . . so it will stretch along one direction and shrink along the other (locally) -- and let's follow the state space at successive times . . .

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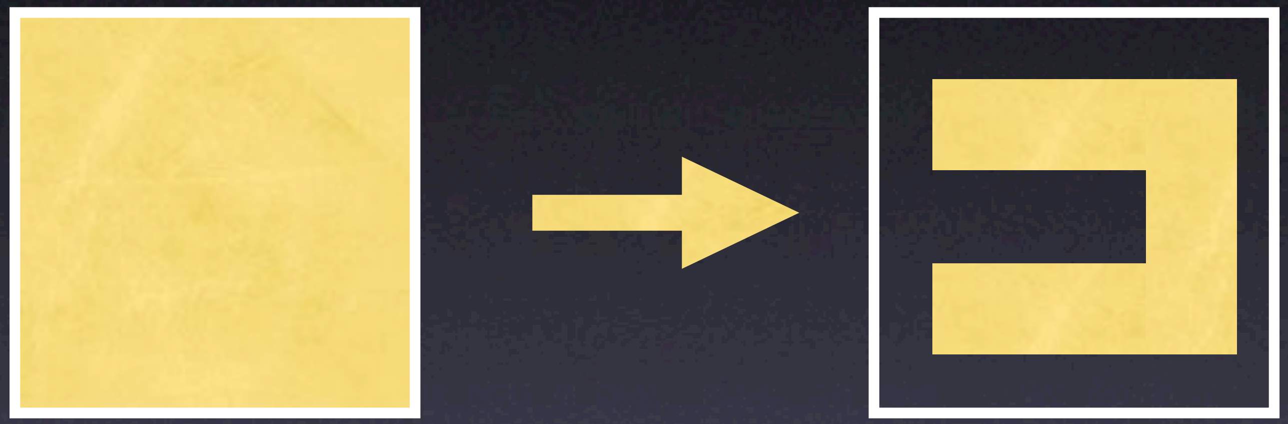

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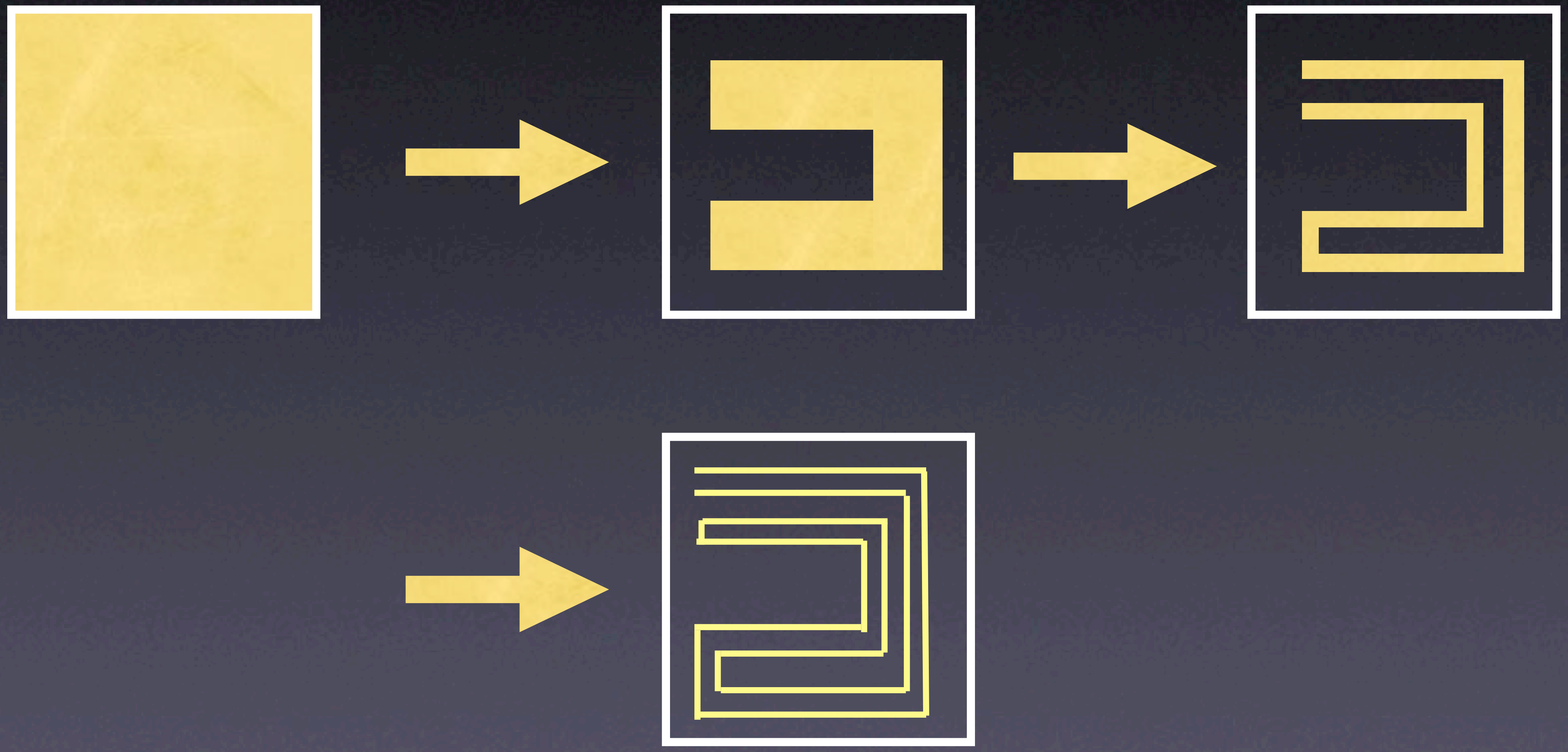

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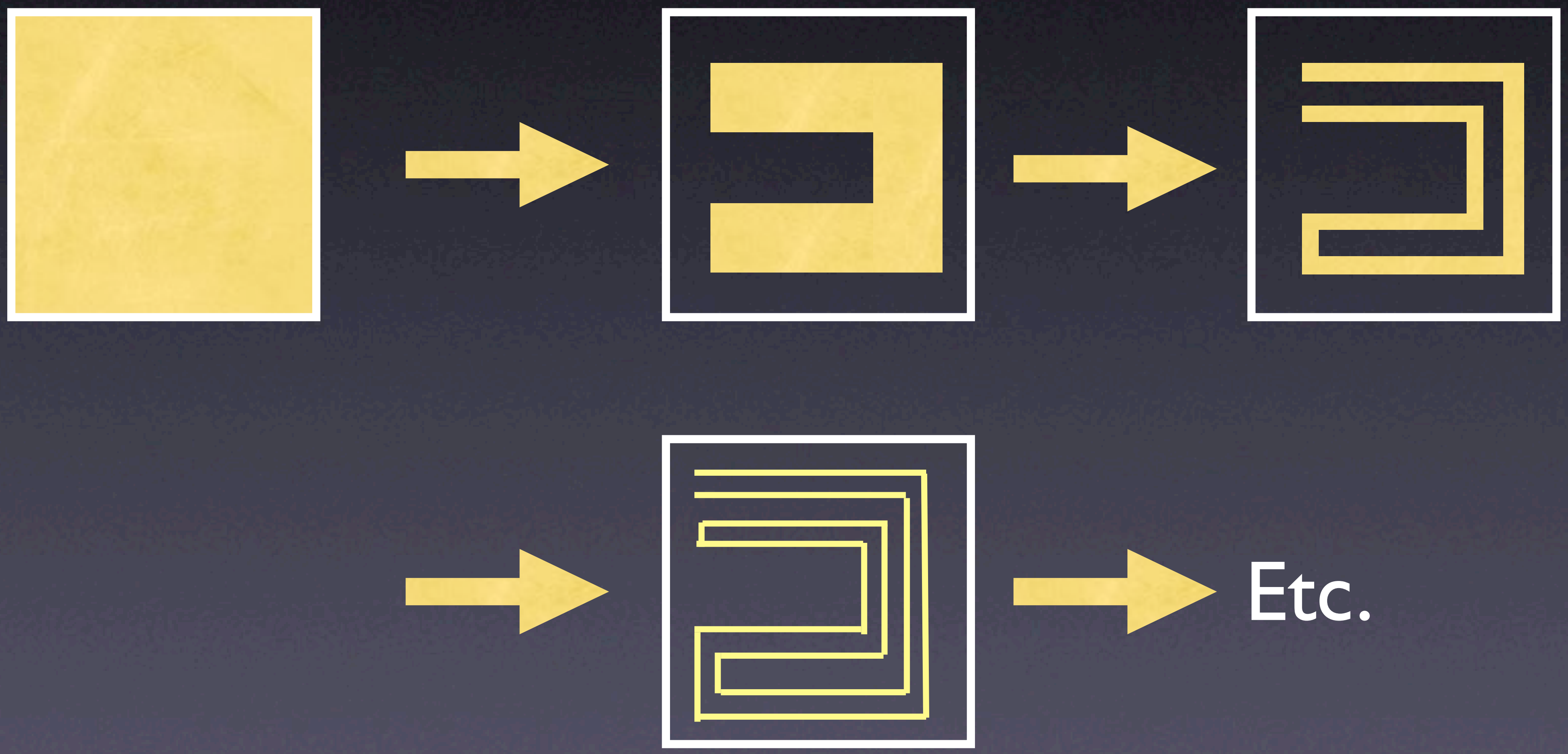
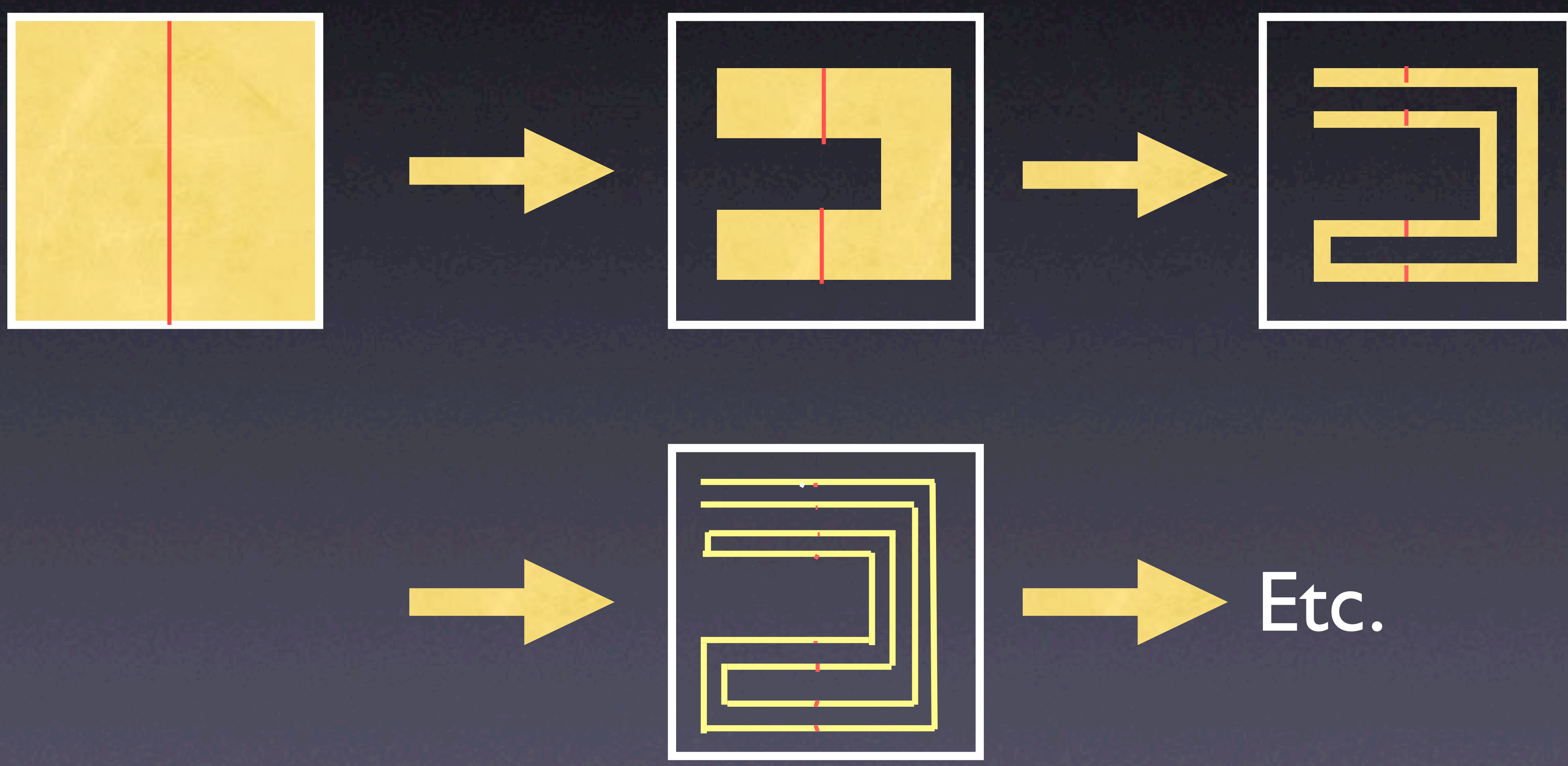


Now let's take a Poincaré section (space slice) through the system



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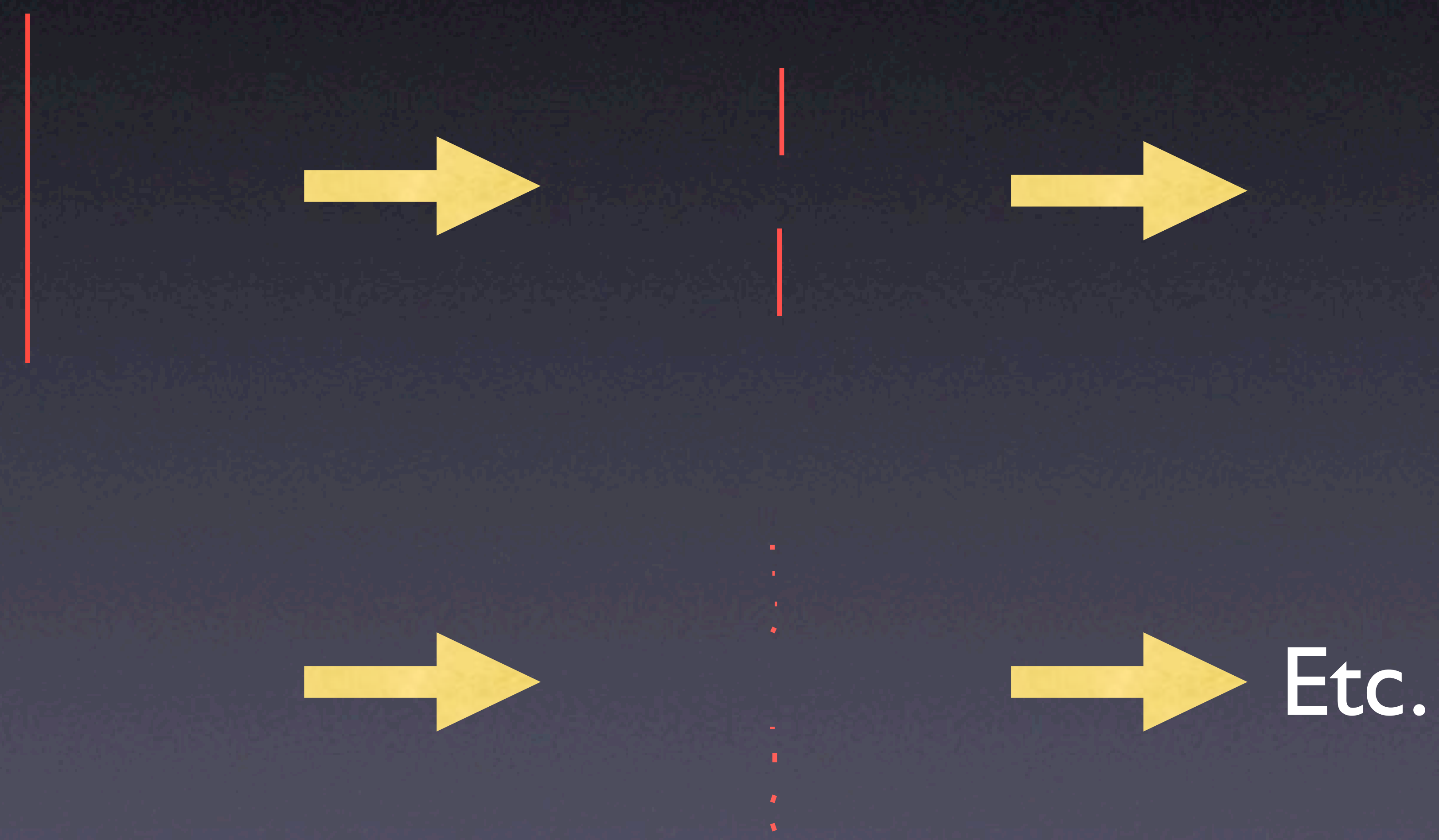

We are building a Cantor set (actually, a slight generalization, a Cantor dust . . .)!

This sort of behavior will be generic for the stretching and folding of the state/phase space for (hyperbolic) dissipative systems . . .

Etc.

So, we should expect to see Cantor dusts in many Poincaré sections of these sorts of systems . . .

Etc.

This also suggests that the fractal dimension of an attractor, or of a Poincaré section of the attractor, can give us the possibility of a characteristic number, to identify the attractor, or at least to distinguish between attractors, and thus between dynamical systems . . .

Fin