Non-Ergodic Dynamics

(for Finance & Economics)

or

Multiplicative Random Walks

Kinnaird Multidisciplinary Research Conference

http://csustan.csustan.edu/~ tom

Tom Carter

Computer Science / Cognitive Studies
CSU Stanislaus

http://csustan.csustan.edu/~
tom/Lecture-Notes/Non-Ergodic/Non-Ergodic.pdf

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Our general topics:

A Simple Example ←

Consider the following gambling game. You are going to bet an amount of money m...

1. You bet your money (say m dollars).

2. A fair coin is fairly flipped
   (i.e., $Pr(heads) = Pr(tails) = 0.5$)

3. If the coin comes up heads, you are paid back $m * 1.50$;
   if the coin comes up tails, you are paid back $m * 0.60$.

Should you be willing to play this game?

Let's calculate the *expected value* of the game:

$$
\begin{aligned}
< game > &= Pr(heads) * m * 1.50 + Pr(tails) * m * 0.60 \\
&= 0.5 * m * 1.50 + 0.5 * m * 0.60 \\
&= m * 0.75 + m * 0.30 \\
&= m * 1.05
\end{aligned}
$$

Thus, by playing the game, you expect to increase the money you wagered by 5%. It seems that you should be happy to play this game (for any amount of money m).

But, let's ask a slightly different question: What is the probability that if you play the game once, you will walk away from the game a winner?

Obviously, if you play the game exactly once, you have a 50/50 chance of walking away a winner (that is, if the coin comes up heads . . .).

On the other hand, in order to actually realize the *expected value*, aren't you likely to have to play the game many times?

Okay, suppose you play the game twice, with an initial bet of \$1.00, and you "let it ride" (in other words, whatever happens in the first coin toss, you bet the entire amount on the next coin flip). The value of the game is: $v(HH) = \$2.25$, $v(HT) = \$0.90$, $v(TH) = \$0.90$, $v(TT) = \$0.36$. So you only have one chance in four of walking away a winner (that is, if the coin comes up heads twice in a row . . .).

Let's explore that some. We'll take this in a particular direction, generalizing into an iterated version. We are going to play the game n times.

But, we're going to do it in a specific way, so that *time* (*history*) matters (we'll come back to this issue later). We'll set things up so that the outcome of later parts of the iteration depend on the results of earlier parts.

The new game follows . . .

1. Let $v(0) = \$1$.

2. Let $j = 0$.

3. Now, repeat n times:

   (a) A fair coin is fairly flipped $(Pr(heads) = Pr(tails) = 0.5)$

   (b) If the coin comes up heads, $v(j+1) = 1.5 * v(j)$.

   (c) if the coin comes up tails, $v(j+1) = 0.6 * v(j)$.

   (d) Set $j = j + 1$.

4. You walk away with $v(n)$ dollars.

Is this a good game to play?

Let's look at this several different ways. First, let's see if we can calculate the *expected value* of the game. Suppose that we play the game for some value of n. There are 2^n possible results of the game, each with probability $\left(\frac{1}{2}\right)^n$. The expected value is then:

$$
\begin{aligned}
\langle game(n) \rangle &= \left(\frac{1}{2}\right)^n * \sum_{j=0}^{n} \binom{n}{j} (0.6)^j (1.5)^{n-j} \\
&= \left(\frac{1}{2}\right)^n * (0.6 + 1.5)^n \\
&= \left(\frac{1}{2}\right)^n * (2.1)^n \\
&= \left(\frac{2.1}{2}\right)^n \\
&= (1.05)^n
\end{aligned}
$$

(as we expected :-)

So, for some specific examples:

$$\begin{aligned}
\langle game(10)\rangle &= 1.63\\
\langle game(20)\rangle &= 2.65\\
\langle game(50)\rangle &= 11.47\\
\langle game(100)\rangle &= 131.5\\
\langle game(1000)\rangle &= 1.55 * 10^{21}
\end{aligned}$$

That means that if I offer you the chance to play $game(100)$, you should be willing to wager $130 (the expected value of the game is greater than your wager) . . .

But, let's ask a slightly different question. What is the most likely amount you will walk away with if you play $game(n)$?

The probability distribution we are working with here is the *binomial distribution* with $p = \frac{1}{2}$.

The value of $game(n)$ if the coin comes up tails j times (and heads the other $n - j$ times) is

$$V(game(n), j) = (0.6)^j * (1.5)^{n-j}.$$

For even n, the *mode* of the distribution happens at $j = \frac{n}{2}$. Hence, the most likely amount you will walk away with will be

$$VML(game(n)) = (0.6)^{n/2} * (1.5)^{n/2} = (0.6 * 1.5)^{n/2} = (0.9)^{n/2}.$$

That says:

$$\begin{aligned}
VML(game(10)) &= 0.35 \\
VML(game(20)) &= 0.12 \\
VML(game(50)) &= 0.005 \\
VML(game(100)) &= 0.00003 \\
VML(game(1000)) &= 1.75 * 10^{-46}
\end{aligned}$$

Hmmm . . . The expected value is going up exponentially, but your most likely return is declining exponentially, and, in fact, faster than the expected value is increasing.

Are you still willing to play the game?

Let's ask another question. Suppose I offer to let you play the game for $1.00. What is the probability that you will walk away from the game having broken even or better?

First, we can look at the number of *tails* you can have and still break even. This will happen when

$$1 = (0.6)^j * (1.5)^{n-j}$$

or (taking logs)

$$j * \log(0.6) + (n - j) * \log(1.5) = 0$$

and so

$$j * (\log(1.5) - \log(0.6)) = n * \log(1.5)$$

and thus

$$j = n * \frac{\log(1.5)}{\log(1.5) - \log(0.6)}$$

This works out to $j \approx 0.4425 * n$.

You will thus break even or better if $j \leq 0.4425 * n$ (remember, j is the number of tails – you can't afford to have too many of them – they're bad ...). The probability of this happening will be

$$prob(j \leq 0.4425 * n) = \left(\frac{1}{2}\right)^n * \sum_{i=0}^{0.4425*n} \binom{n}{i}$$

So, how can we calculate that?

One estimate that we can use, coming from Hoeffding's inequality, is

$$prob(j \leq k) \leq \text{exp}\left(-2 * \frac{(\frac{n}{2} - k)^2}{n}\right)$$

Putting in $k = 0.4425 * n$, we have, for example, when $n = 1000$:

$$\begin{aligned} prob(j \leq 443) &\leq \text{exp}\left(-2 * \frac{(500 - 442)^2}{1000}\right) \\ &= \text{exp}\left(-2 * \frac{58^2}{1000}\right) \\ &= \text{exp}\left(-2 * \frac{3364}{1000}\right) \\ &= \text{exp}\left(-2 * 3.364\right) \\ &= \text{exp}\left(-6.728\right) \\ &\approx 0.0012 \end{aligned}$$

Just for fun, here is another estimation process derived from Stirling's approximation for the factorial for $\binom{n}{i} = \frac{n!}{i!(n-i)!}$.

As noted, we have

$$prob(j \leq \lambda * n) = \left(\frac{1}{2}\right)^n * \sum_{i=0}^{\lambda * n} \binom{n}{i}$$

We can make an estimate of this as follows, for $\lambda < \frac{1}{2}$:

$$\left(\frac{1}{2}\right)^n * \sum_{i=0}^{\lambda * n} \binom{n}{i} \leq 2^{-n} * 2^{nH_2(\lambda)} = 2^{n(H_2(\lambda)-1)}$$

where $H_2(\lambda) = \lambda * \log_2(\frac{1}{\lambda}) + (1 - \lambda) * \log_2(\frac{1}{1-\lambda})$.

Using $\lambda = 0.4425$, we have

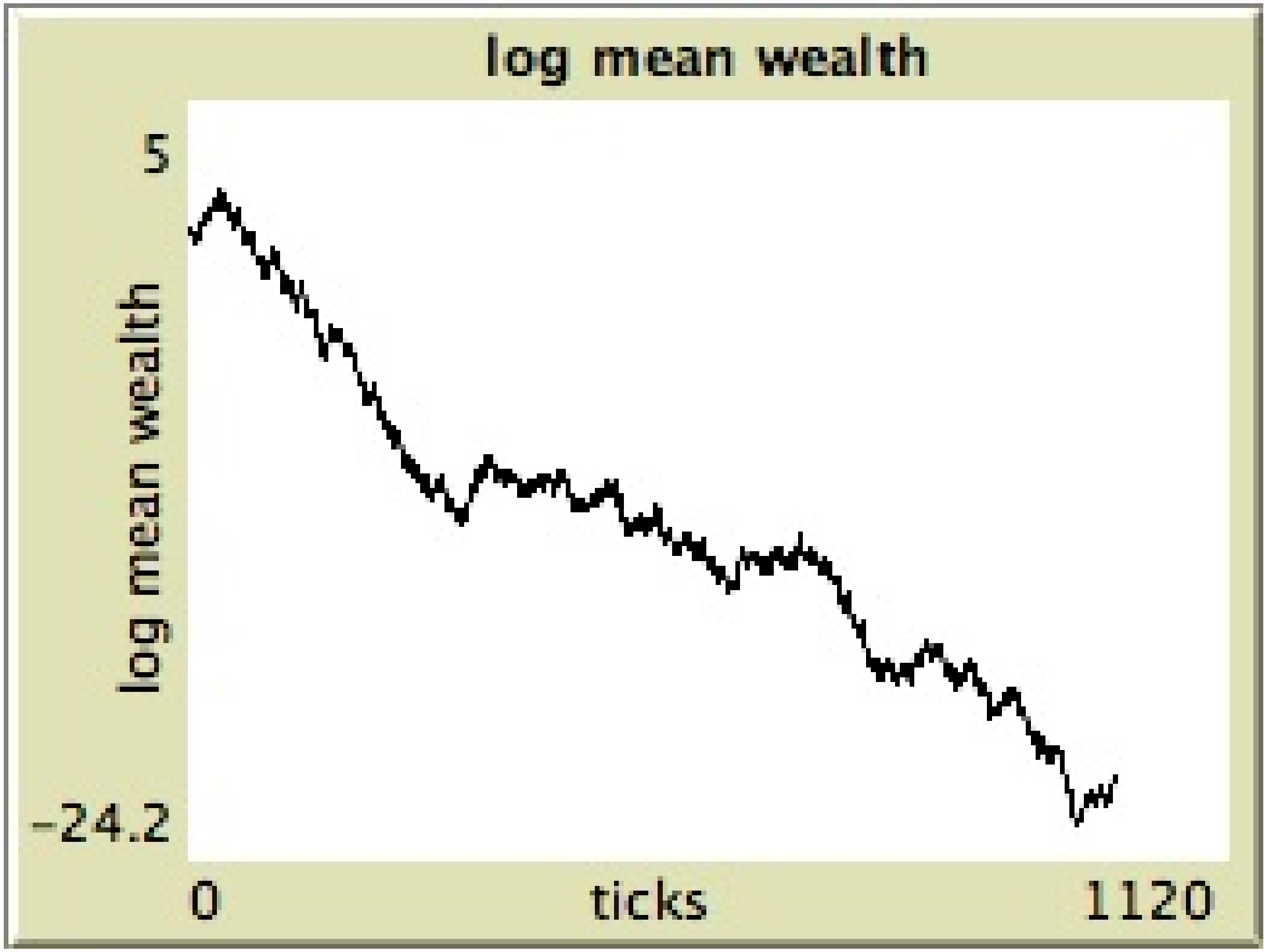
$$H_2(0.4425) - 1 = 0.99044 - 1 = -0.00956$$

This gives us the estimate (essentially the same as above):

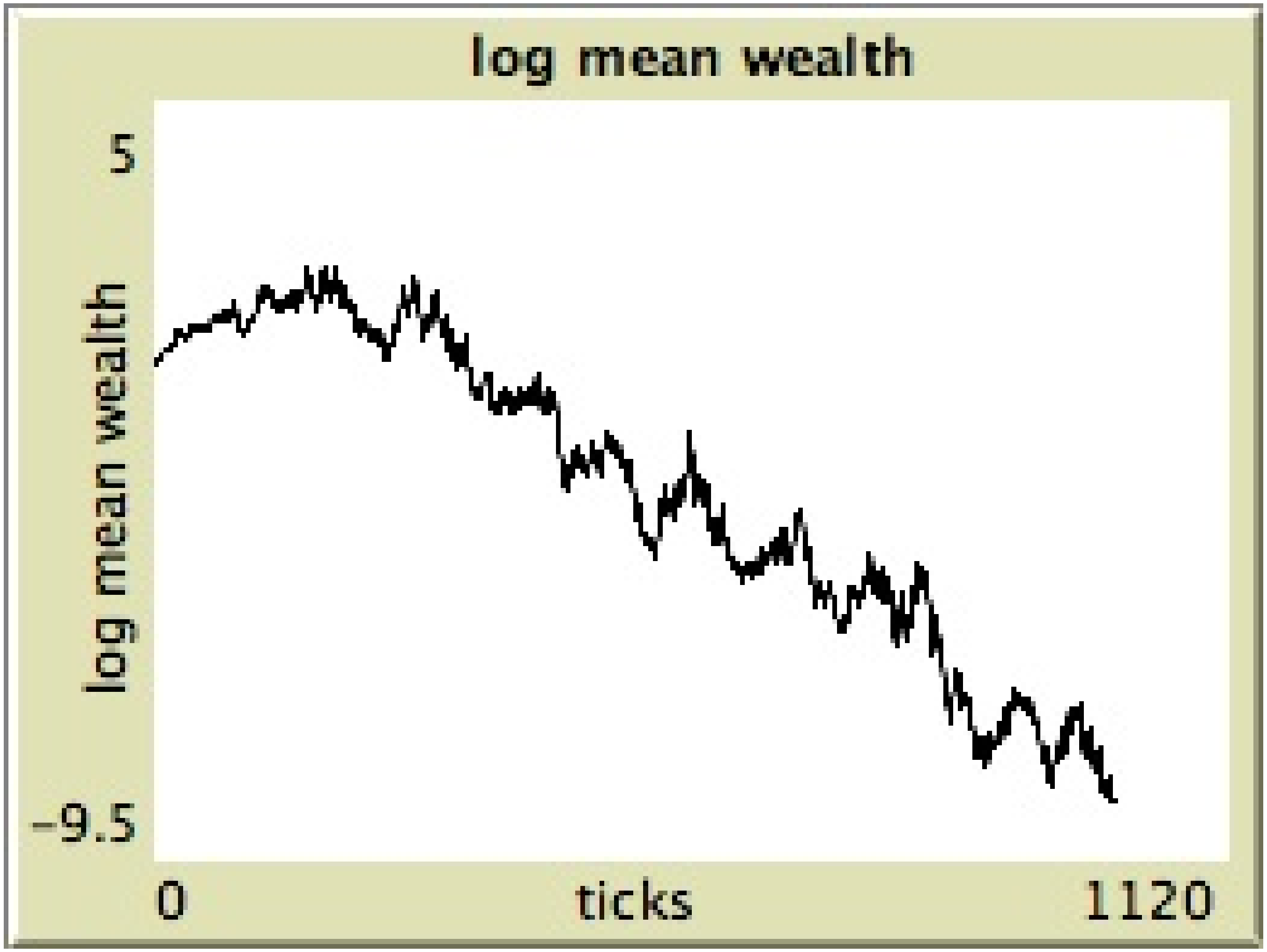
$$prob(j \leq 0.4425 * 1000) \leq 2^{-9.56} = 0.00132.$$

This means that you only have about 1 chance in 1000 of walking away a winner. Now do you want to play the game?

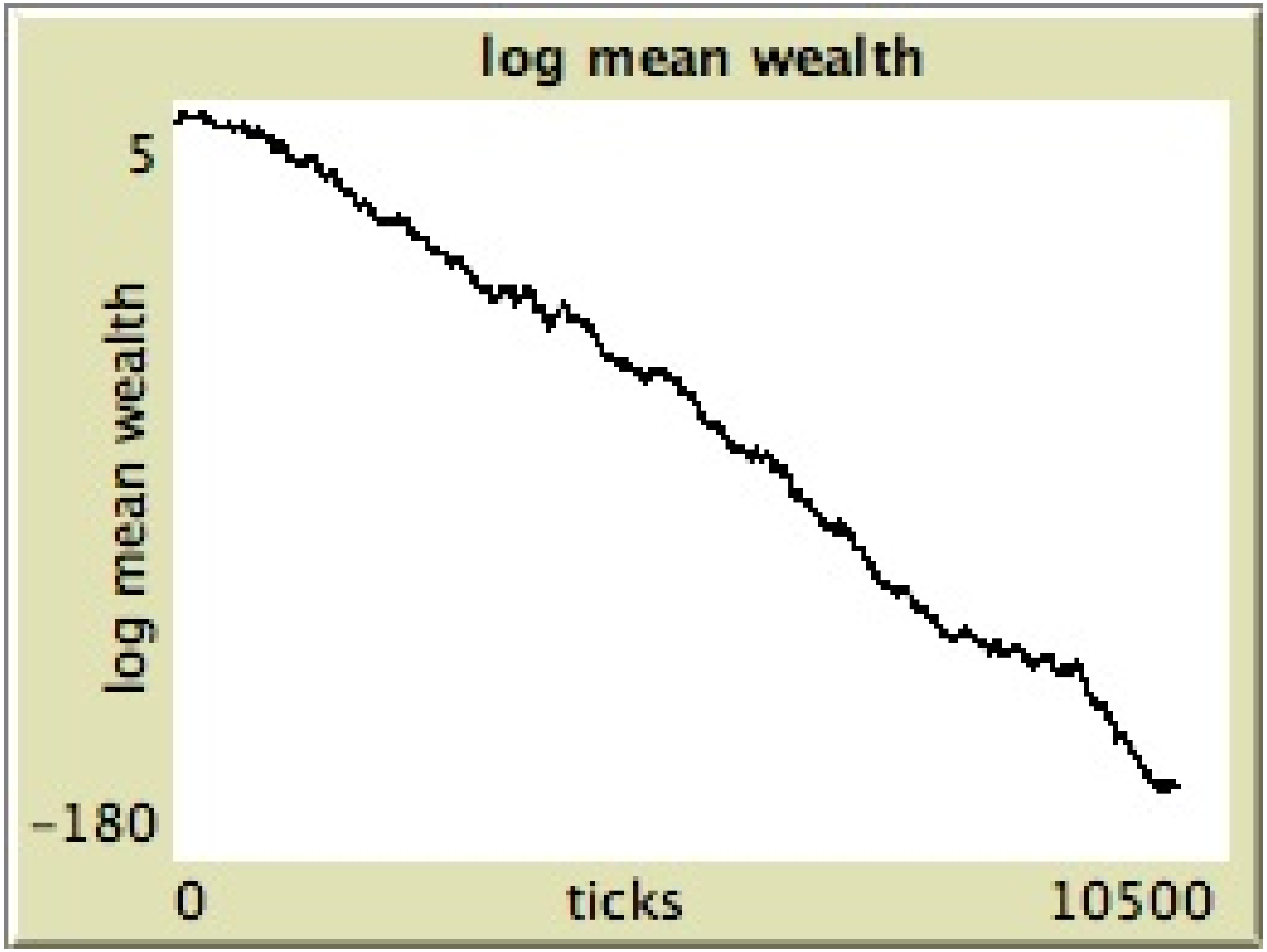
To get a sense of what happens, let's look at examples of the results of playing the "1000 steps" game. Following are some examples. In the first example, one player plays the "1000 steps" game. Then we can see an example where we look at the mean results of 500 players each playing the "1000 steps" game, and finally 2000 players each playing a "10,000 steps" game. These plots are from a NetLogo model (that you use to explore these issues . . . see link in the references). The graphs are log-linear – thus helping to see "exponential change" over time.



Playing the 1.5 / 0.6 game for 1000 steps (NetLogo model)



500 players playing the 1.5 / 0.6 game for 1000 steps (NetLogo model)



2000 players playing the 1.5 / 0.6 game for 10000 steps (NetLogo model)

We can see that despite the fact that the expected value of the game at each step is 1.05 (which is > 1), nonetheless, this is not a good game to play many times over (assuming that you "let it ride" each time).

Feeling lucky?

All right – let's play a slightly different game. The general setup is the same, except instead of multiplying your wealth at each step by either 1.5 (on heads) or 0.6 (on tails), we'll draw your "multiplier" from a normal distribution with mean μ and variance σ^2. In other words, we'll play the game with

$$v(j+1) = v(j) * X_j$$

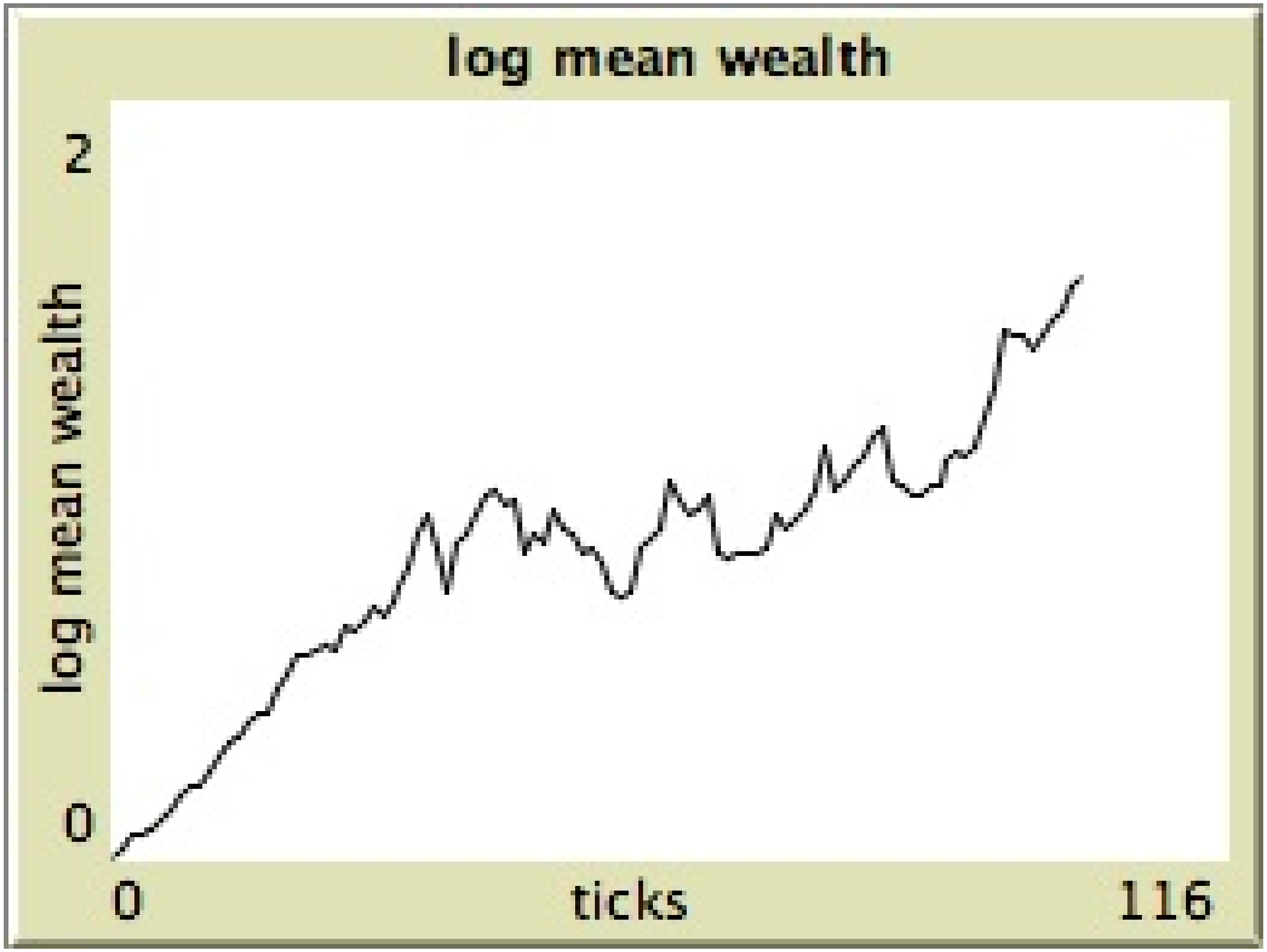
where $X_j \in Normal(\mu, \sigma^2)$, and you "let it ride" n times.

Actually, I'll do a little better than that . . . I'll put a "floor" on the multiplier X: if $X < 0.001$, I'll replace X by 0.001 before I multiply. Note that this will keep your "winnings" from ever going negative . . .

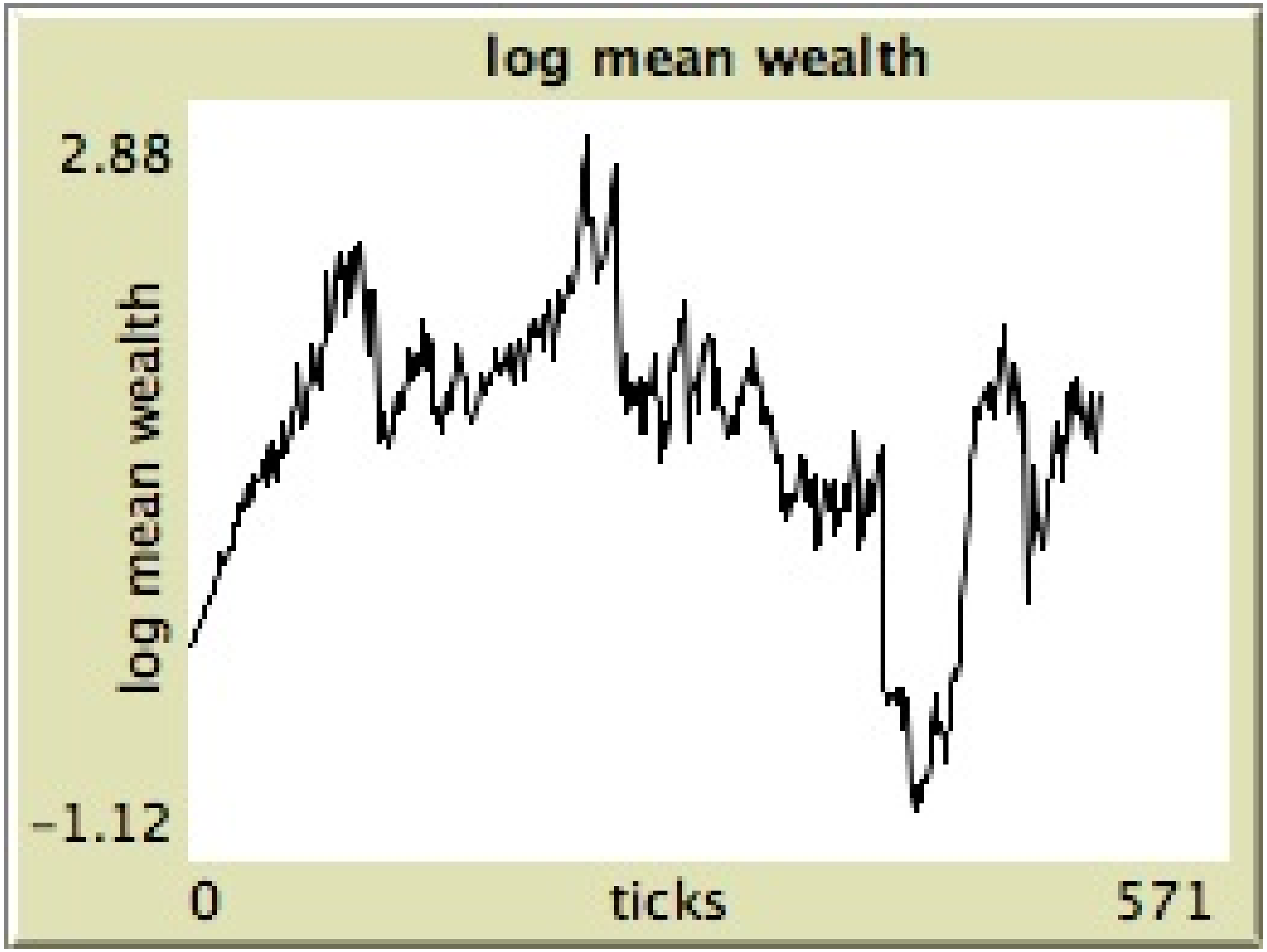
Should you be willing to play this game for, say, $n = 1000$ with $\mu = 1.05$ and $\sigma^2 = 0.15$?

I'll let you do some "estimated value" calculations.

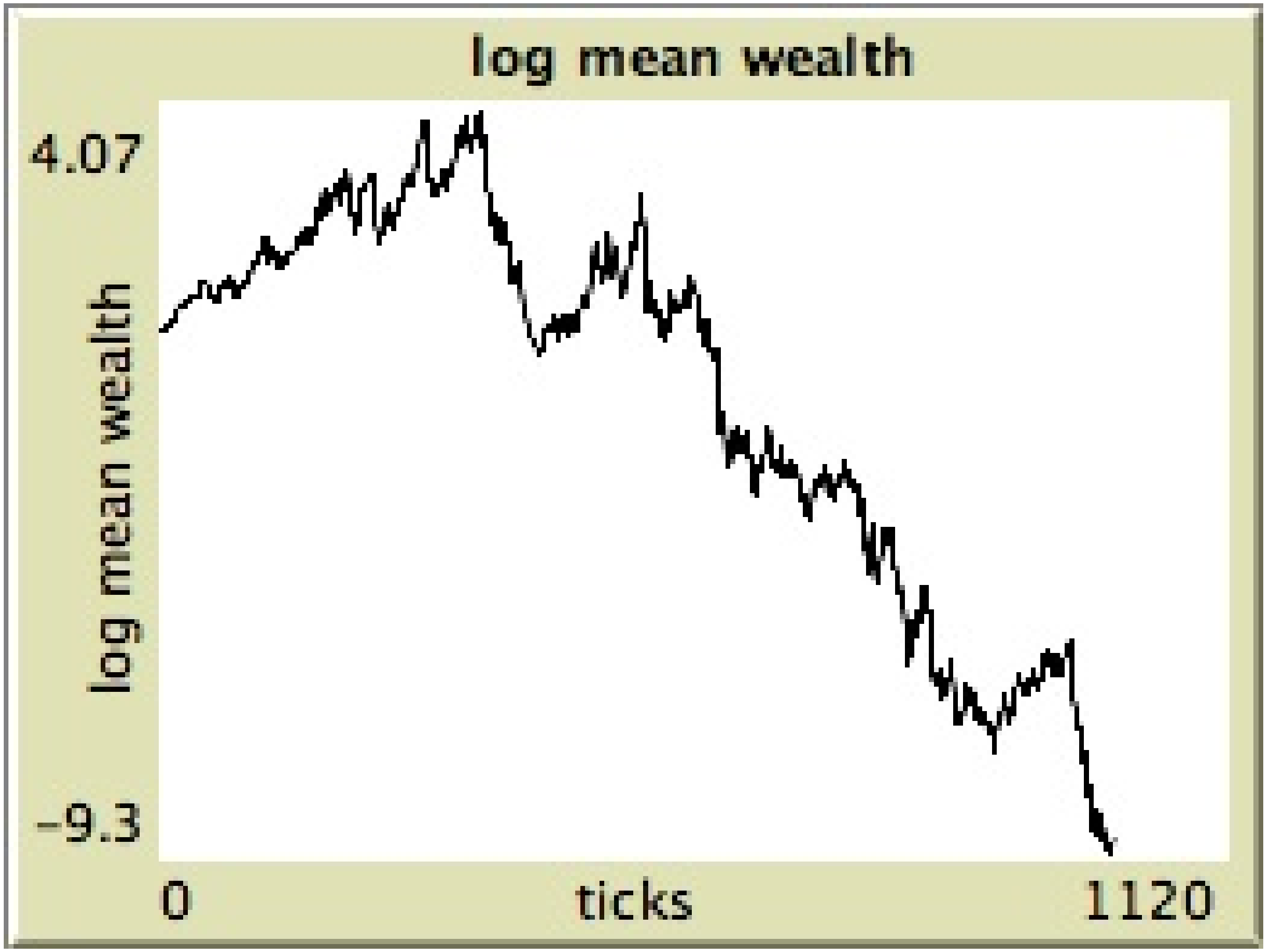
But, following are some reasonably typical runs (in NetLogo) of this, for 100, 500, 1000, and 10000 step games. In each case, we are averaging 300 players' results. Again, these are from NetLogo, and are log-linear plots.



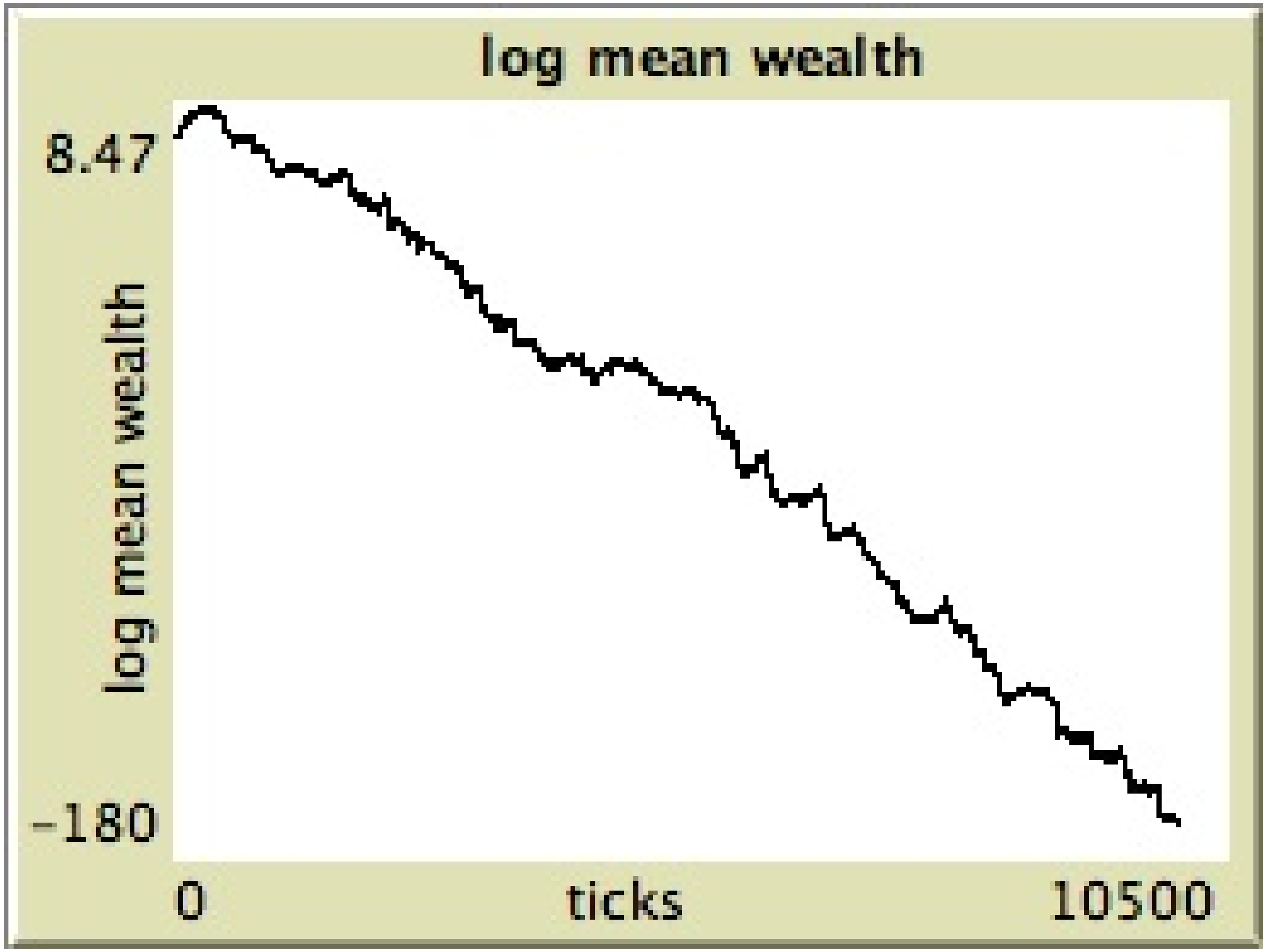
100 step game, with $X \in Normal(1.05, 0.15)$, averaging 300 players.



500 step game, with $X \in Normal(1.05, 0.15)$, averaging 300 players.



1,000 step game, with $X \in Normal(1.05, 0.15)$, averaging 300 players.



10,000 step game, with $X \in Normal(1.05, 0.15)$, averaging 300 players.

For 100 steps, things look pretty good – the return seems to be growing exponentially (remember, the graphs are log-linear, so a straight line is exponential growth – or decay . . .). But, at 500 steps, things are starting to flop around. By the time we get to 1,000 steps, things are looking pretty bad. And by 10,000 steps, we are clearly declining exponentially.

Ergodicity ←

So, what does this have to do with *ergodicity*?

In general terms, a dynamical system is called *ergodic* if the average over time of the system is equal to the "space" (ensemble) average of the system. In other words, if a system is *ergodic*, we can follow the trajectory of a single example (realization) of the system in order to explore the whole space of possible behaviors of the system.

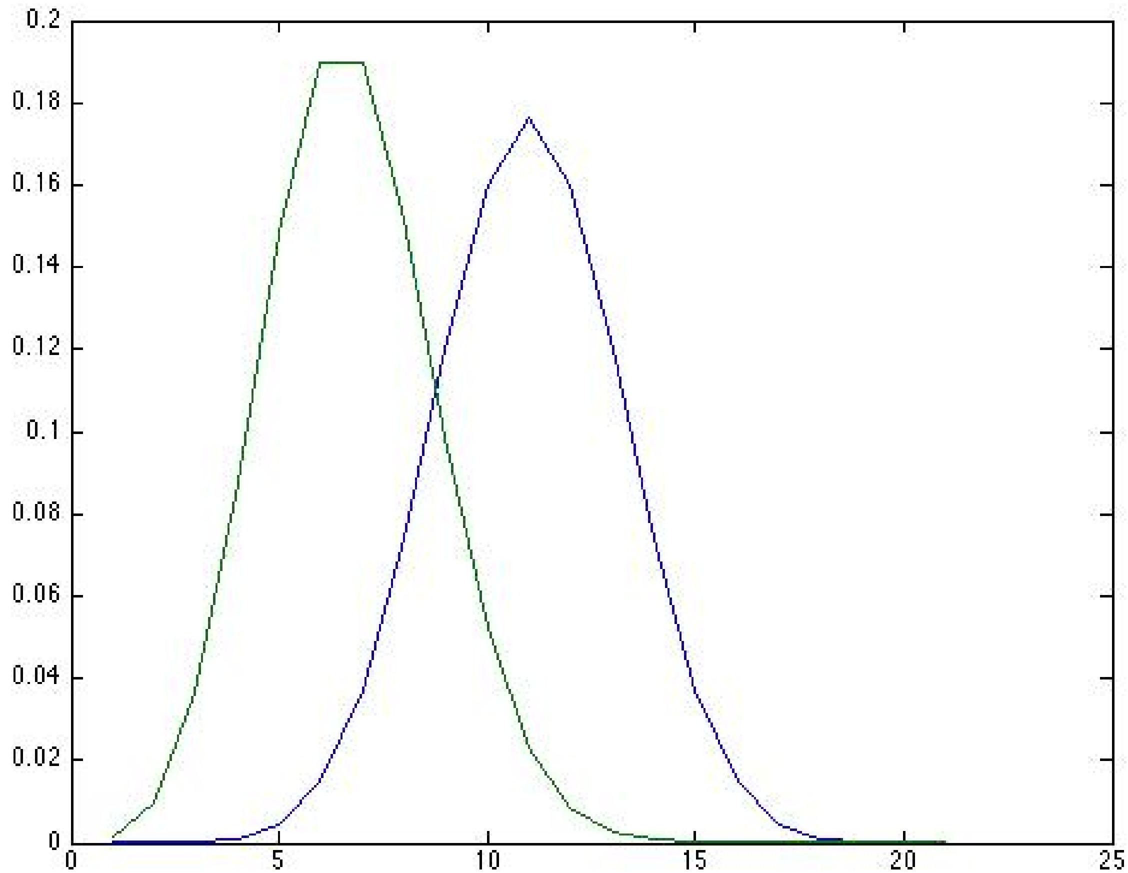
If a system is ergodic, we should be able to learn about the dynamics either by theoretical analysis of the ensemble behavior of the system, or by tracking a relatively small number of individual trajectories. In the examples we have been checking, trying to predict the long term results (trajectories) by calculating *estimated values* in effect assumes that the systems are ergodic – in other words, that an individual trajectory will explore enough of the space of possibilities to approximate the expected ensemble value.

What we have seen is that in our examples, *expected values* do not do a good job of predicting individual trajectories. In fact, even following "many" (hundreds?) of trajectories does not reflect the ensemble average expected value.

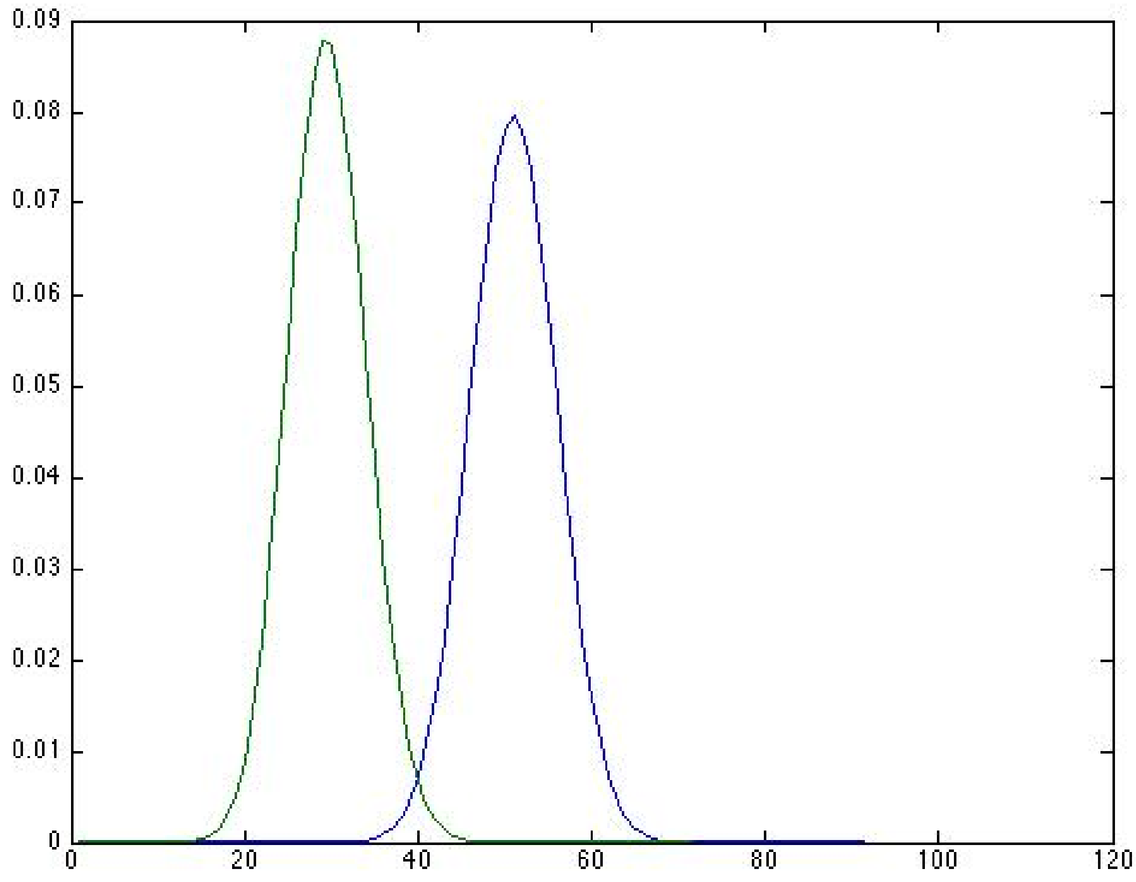
In the heads/tails (1.5 / 0.6) multiple steps games, larger payoffs are skewed to smaller numbers of tails. In order to be a winner, you need to have enough more heads than tails (i.e., something less than 45% tails). You do have a chance of being a large winner (e.g., if you got 1,000 heads in a row in the "1,000 steps" game, you would win something like $\$10^{176}$ – but of course the probability of that happening would be $2^{-1000} \approx 10^{-301}$).

An important issue is the role of *time* (*history*) in the dynamics. For example, if you are playing the 1,000 step game, and in the first 100 steps you have seen 50 tails, there is no chance that later in the game your trajectory will explore portions of the space having fewer than 50 tails. And, in general, each time you see a tails, more of the "good" part of the space of possibilities is cut off from future exploration. Thus, the history of the trajectory matters, and the system is not *ergodic*. This means that the *ensemble average* (i.e., the *expected value*) is not a good estimator of the results of actually playing the "1,000 steps" game.

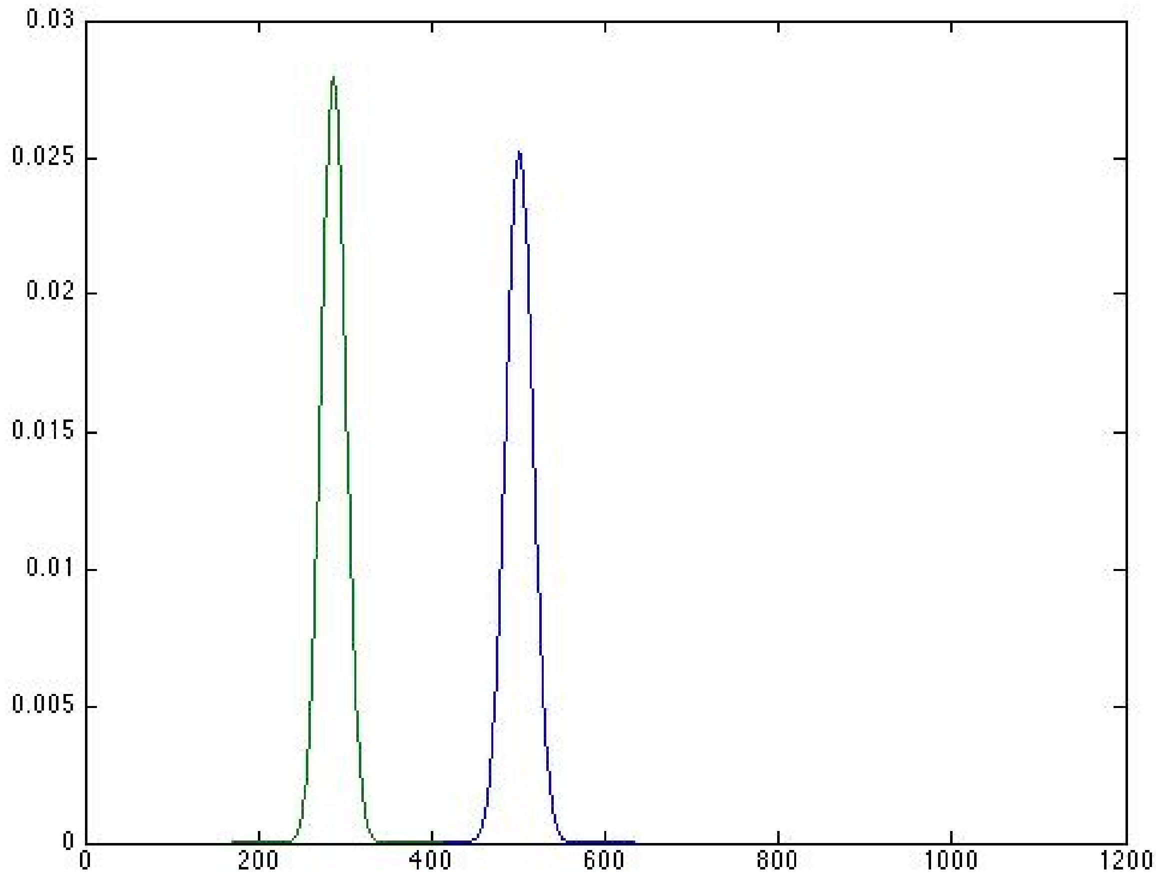
On the next few pages are some plots that show the relationship of the binomial distribution to the "expected value" concentrations. To see things better, the concentration of "expected value" is "normalized".



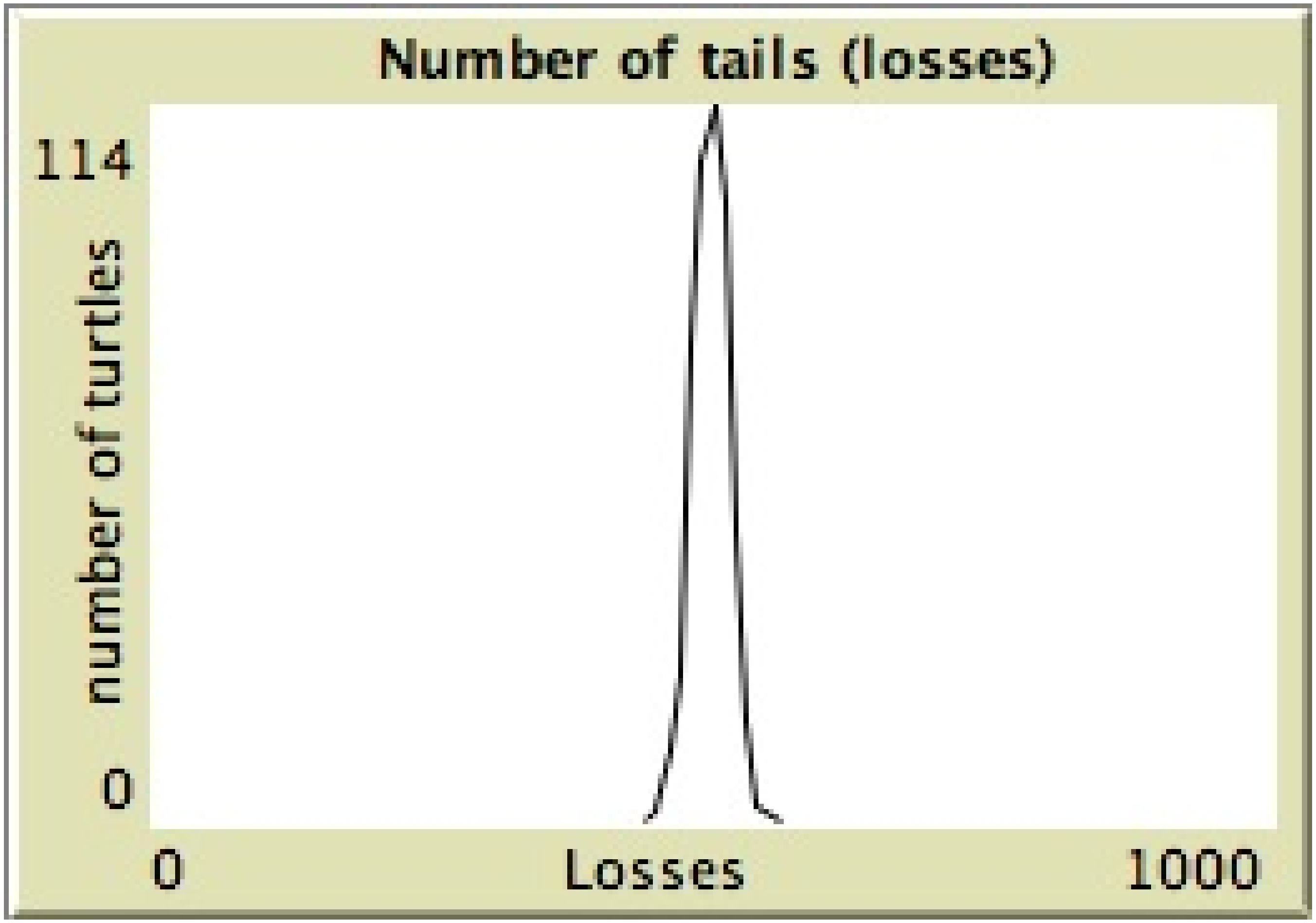
Binomial distribution ($p = \frac{1}{2}$, $n = 20$)(blue) and "normalized concentration of expected value" $(1.5, 0.6)$ (green)



Binomial distribution ($p = \frac{1}{2}$, $n = 100$)(blue) and "normalized concentration of expected value" $(1.5, 0.6)$ (green)



Binomial distribution ($p = \frac{1}{2}$, $n = 1000$)(blue) and "normalized concentration of expected value" $(1.5, 0.6)$ (green)



histogram of number of tails (losses) for 500 players, each playing the "1000 step" game (NetLogo model)

We can see that for relatively "small" values of n, the n-steps game will be in a realm where there is reasonable "overlap" between the central tendency of the binomial distribution and the "concentration of expected value." But, as n gets larger, there is separation between the two. Thus, at the beginning, the "expected value" will be a reasonable estimate of the return from the game. But, at some point, the non-ergodicity will take over, and the history of the trajectory will win out.

By averaging the trajectories of many "players", we can increase the likelihood that at least some players will (for a while, . . .) explore the more favorable regions of the space of possibilities. But, eventually, the *law of large numbers* will take over, and with overwhelming probability, things will go badly.

More General Analysis ←

One issue that we might consider is the effect of the precise values (1.5 and 0.6) that we have chosen to explore. In fact, those values were chosen with some malice aforethought, to make the demonstration / pedagogy more effective.

In particular, the values were chosen so that (at least), the *expected value* was greater than 1 ($0.5 * 1.5 + 0.5 * 0.6 = 1.05$), but the product of the two was less than 1 ($1.5 * 0.6 = 0.9$).

The "normal distribution multiplier" game is somewhat more general, and thus perhaps amenable to some more careful analysis.

I will also note that mathematical history, and the propensity of researchers, are such that in fact, we are likely to find better tools for analysis if we move to a continuous version of the system, rather than a discrete, step-by-step version. We'll come back to this issue later.

So, for a while, let's explore the $game(\mu, \sigma^2, n, s)$ system where μ and σ^2 are the mean and variance of the normal distribution from which the multipliers are taken, n is the number of steps in the game, and s is the number of different trajectories we will follow (typically, we will average over the s trajectories . . .).

The "value" of the game is thus

$$V(game(\mu, \sigma^2, n, s)) = \frac{1}{s} \sum_{i=1}^{s} \left(\prod_{j=1}^{n} X_j \right)$$

where $X_j \in normal(\mu, \sigma^2)$ are independent, identically distributed random variables.

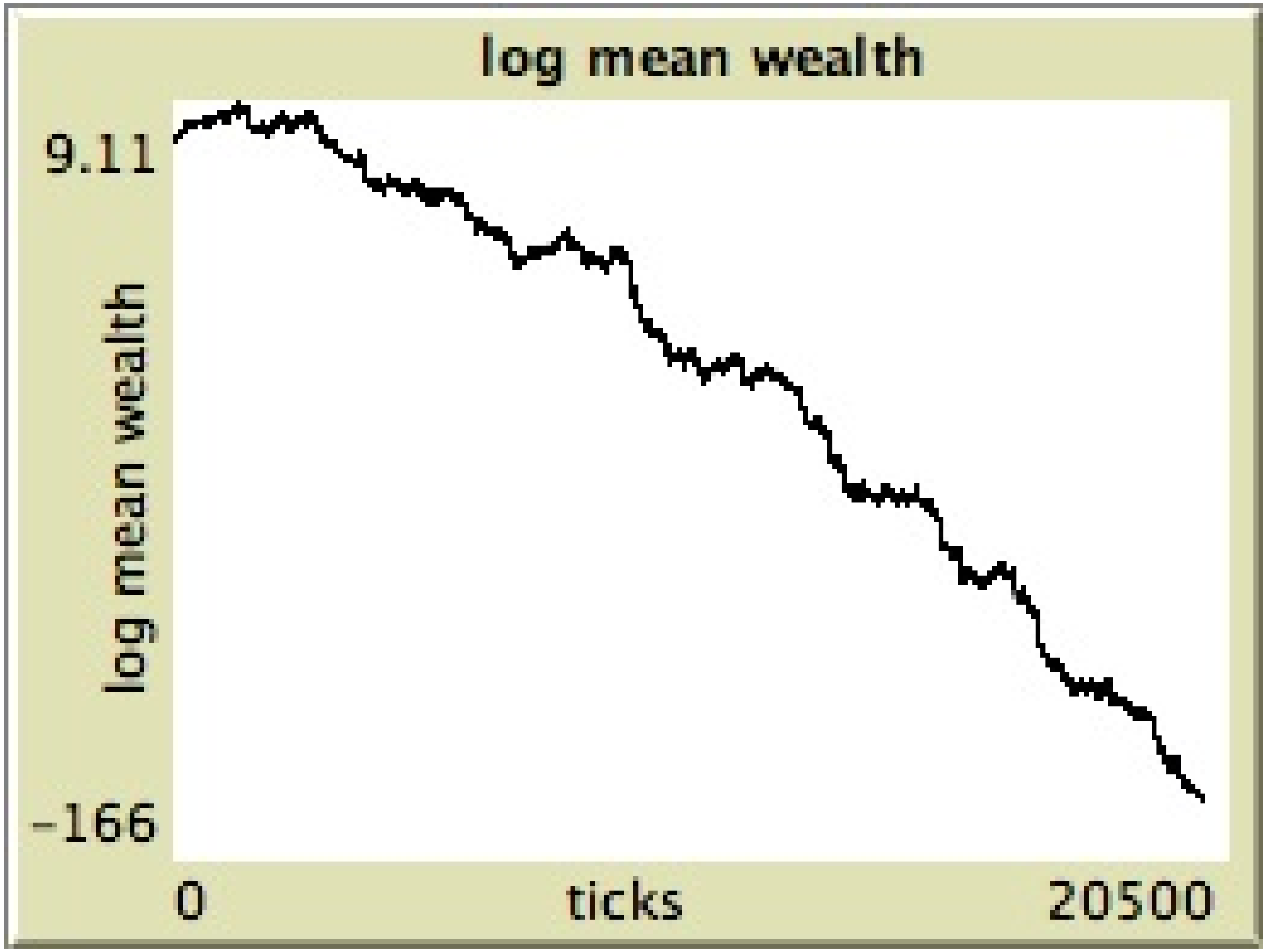
We are interested in exploring the interactions among the various parameters.

Following are several examples. In particular, they are examples of

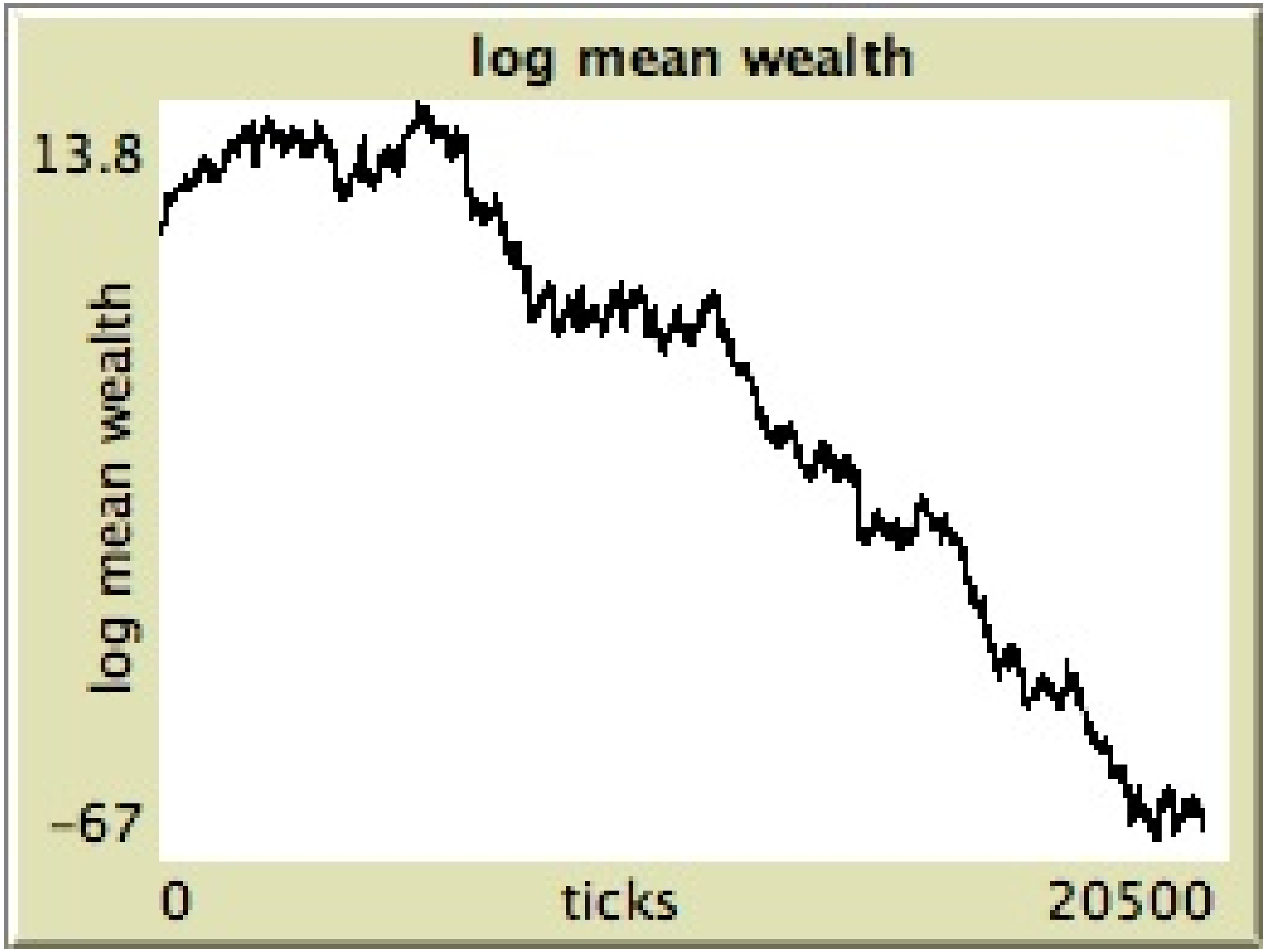
$$game(\mu, \sigma^2, n, s)$$

with $\mu = 1.05$, $n = 20,000$, $s = 2000$, and $\sigma^2 = 0.12, 0.11, 0.10,$ $0.09, 0.08$.

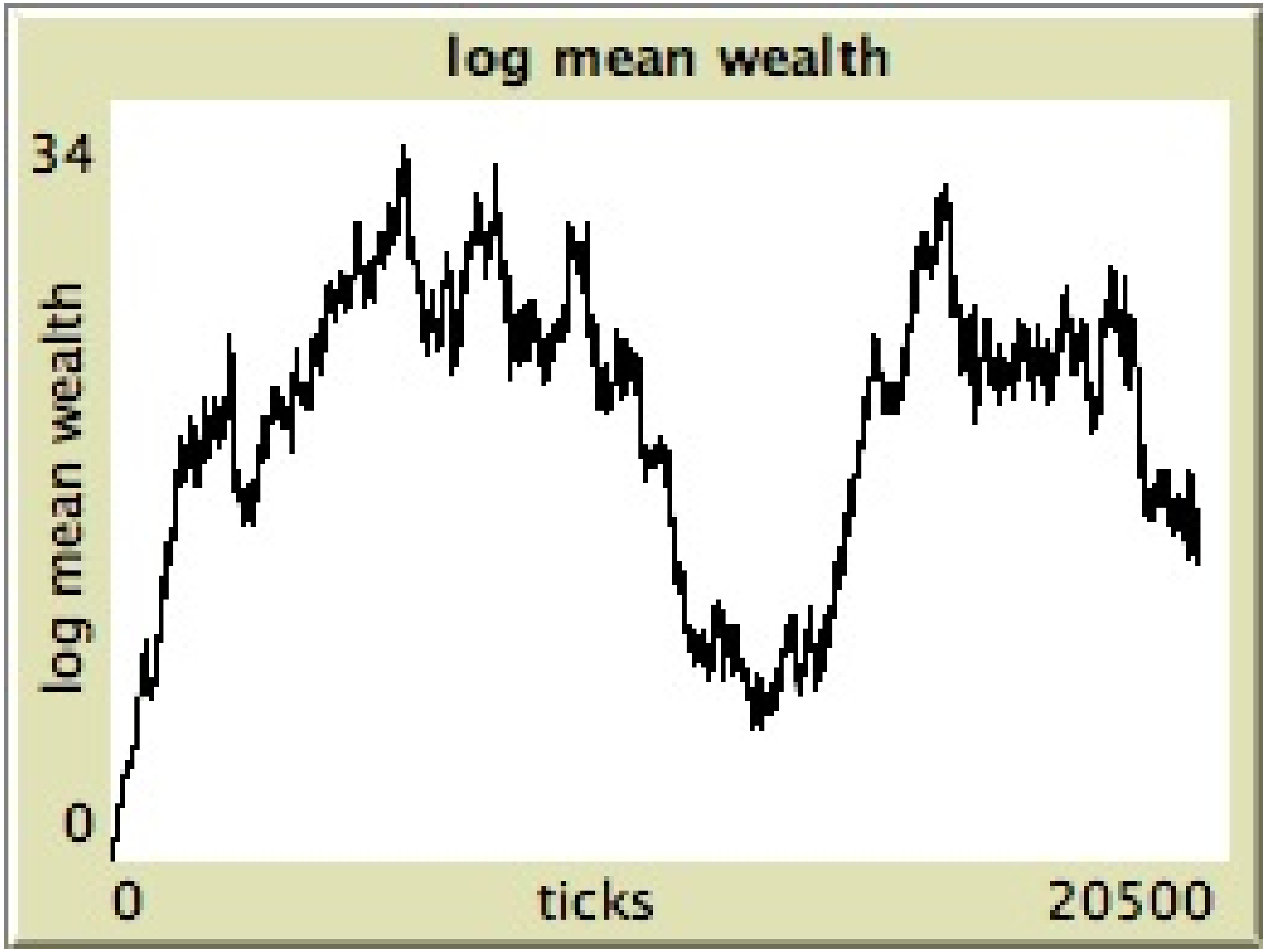
Thus, we are doing a parameter sweep on σ^2, but keeping the other parameters constant. There are some interesting differences for the various values of σ^2.



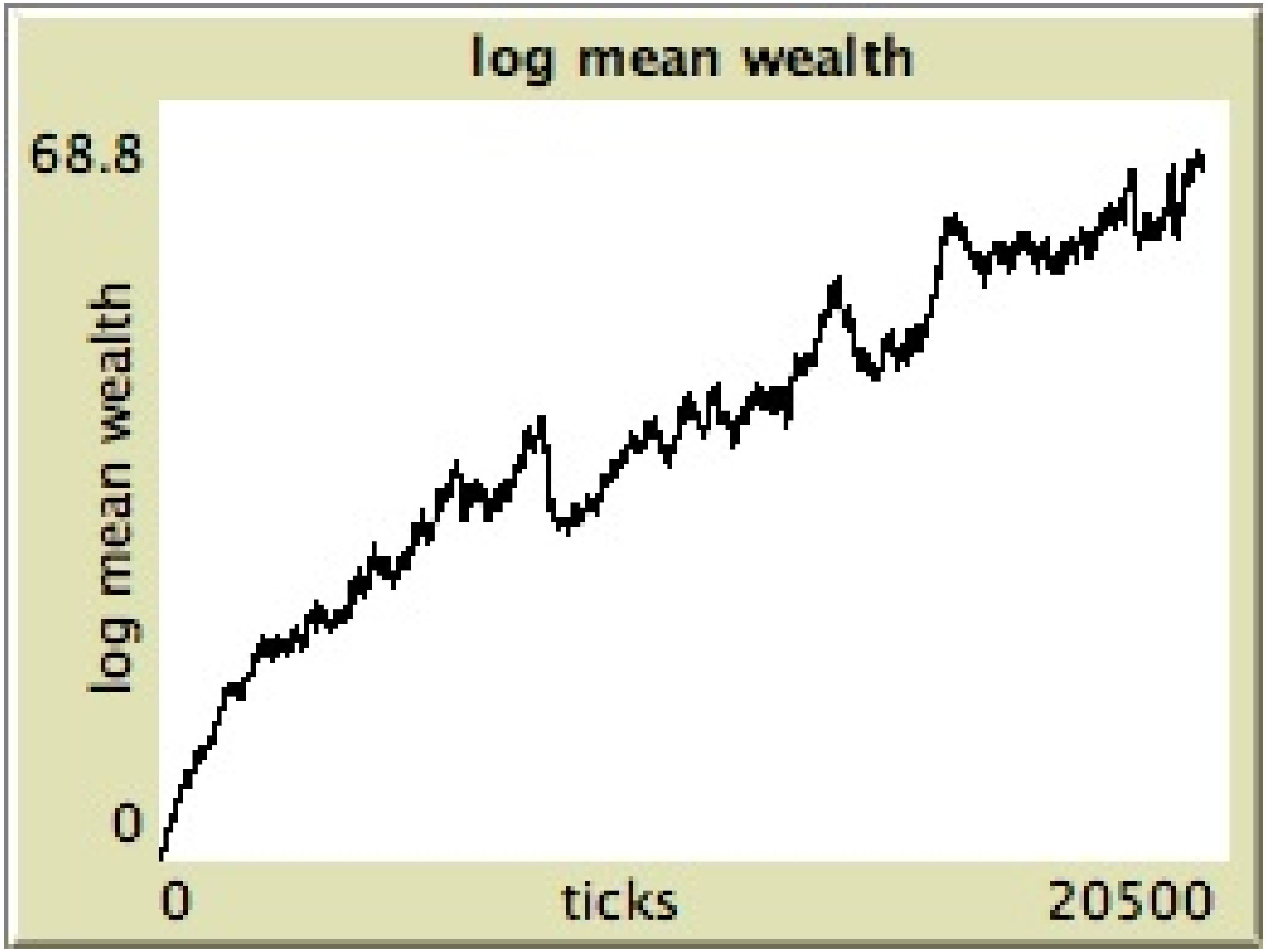
2000 players playing the "normal" $\mu = 1.05, \sigma^2 = 0.12$ for 20,000 steps game (NetLogo model)



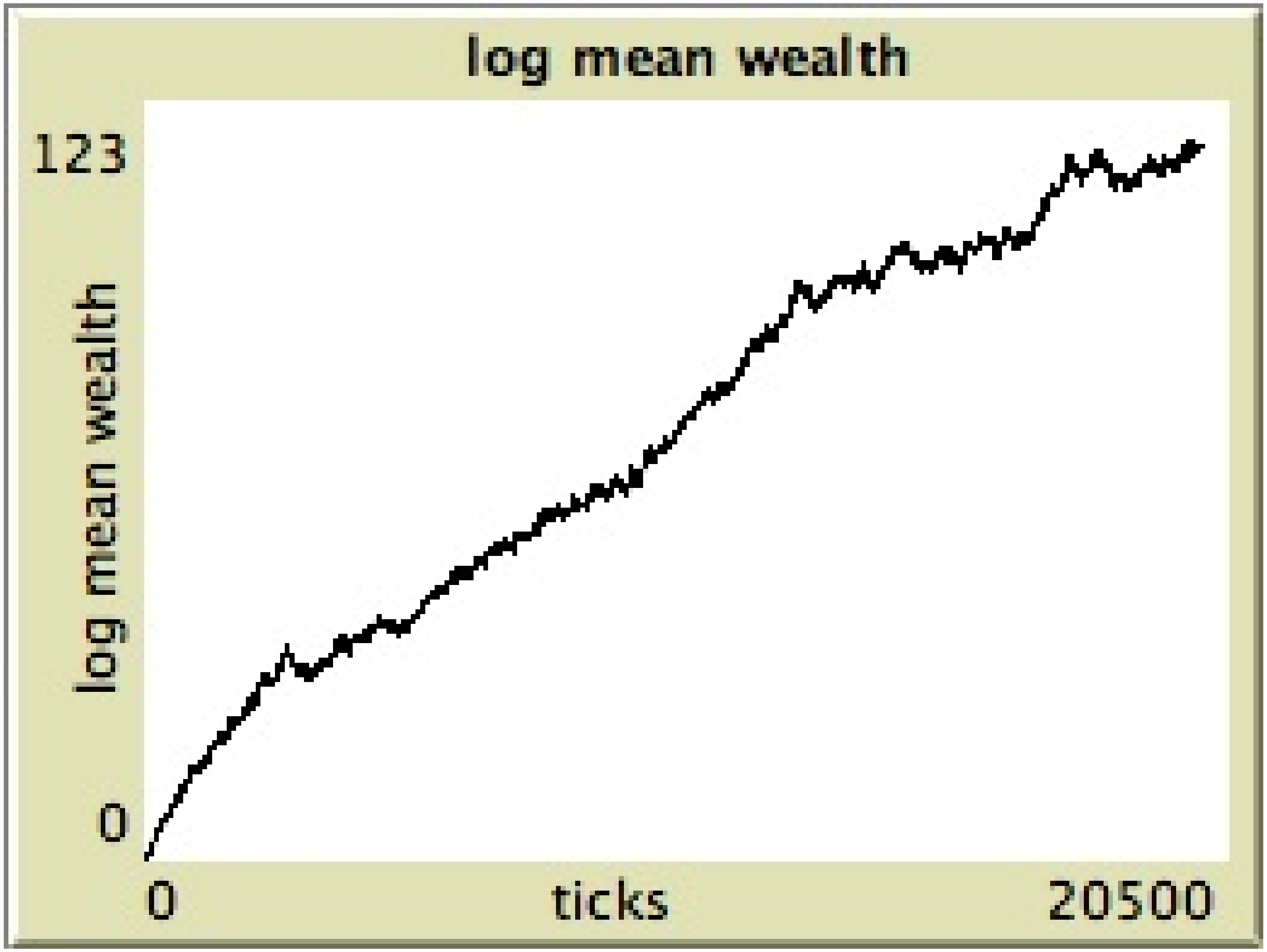
2000 players playing the "normal" $\mu = 1.05, \sigma^2 = 0.11$ for 20,000 steps game (NetLogo model)



2000 players playing the "normal" $\mu = 1.05, \sigma^2 = 0.10$ for 20,000 steps game (NetLogo model)



2000 players playing the "normal" $\mu = 1.05, \sigma^2 = 0.09$ for 20,000 steps game (NetLogo model)



2000 players playing the "normal" $\mu = 1.05, \sigma^2 = 0.08$ for 20,000 steps game (NetLogo model)

We won't go into detailed analysis here (see references, below). But, what we find is that the time average of the system is (largely) controlled by

$$\mu - \frac{\sigma^2}{2}.$$

We are mostly interested in μ greater than 1.

When $\mu - \frac{\sigma^2}{2} > 1$, the system (largely) grows exponentially, more or less following the *expected value*, or *ensemble average* of the system.

If $\mu - \frac{\sigma^2}{2} = 1$, the long term system average is roughly constant.

If $\mu - \frac{\sigma^2}{2} < 1$, the system (generally) decays exponentially.

More Multiplicative Random Walks ←

Let's look at this from the perspective of multiplicative random walks. We have, for an individual trajectory of the system

$$v(j+1) = v(j) * X_j$$

where X_j is a random variable. Thus, for a game of n steps, and assuming $v(0) = 1$, we have

$$v(n) = \prod_{j=1}^{n} X_j$$

In the cases we have been exploring, we are assuming the X_j are independent, identically distributed.

What we have, then, is that the $v(n)$ are also random variables. Let's see what we can learn about the distribution of $v(n)$.

One thing we can do is take logs:

$$
\begin{aligned}
\ln(v(n)) &= \ln\left(\prod_{j=1}^{n} X_j\right) \\
&= \sum_{j=1}^{n} ln(X_j)
\end{aligned}
$$

Now, we can see that $ln(X_j)$ are also i.i.d. random variables. If we assume that the variance of $ln(X_j)$ if finite, then a *Central Limit Theorem* will tell us that

$$
\frac{1}{n}\ln(v(n)) = \frac{1}{n}\sum_{j=1}^{n} ln(X_j)
$$

tends to a normal distribution as n grows.

This indicates that we should think in terms of a *log-normal* distribution . . .

The probability density function for a lognormal distribution is:

$$f_X(x;\mu,\sigma) = \frac{1}{x\sigma\sqrt{2\pi}}\, e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}},$$

with $x > 0$.

This gives us some ways to think about the system.

Another kind of approach is to move in the direction of continuous versions of the system, in which case we can expect to work with a *stochastic differential equation* such as

$$dx = x(\mu dt + \sigma dW)$$

where we have a *drift* term μ, σ tells the amplitude of the noise, and

$$W(t) = \int_0^t dW$$

is a *Wiener process*.

The *Wiener process* is, in general, a continuous-time form of Brownian motion.

Things can be somewhat tricky when trying to solve (integrate) stochastic differential equations. In particular, we have the "noise" term dW, and we need to be careful how we understand or interpret or model this term. In general, $W(t)$ is considered to be a continuous but nowhere differentiable function, so we must be thoughtful when doing integration. There are various approaches to such integration – an important approach is often called the *Itō calculus*, after Kiyoshi Itō.

Much more discussion of these topics can be found in the references.

Some Implications ←

These issues of non-ergodicity have some interesting implications.

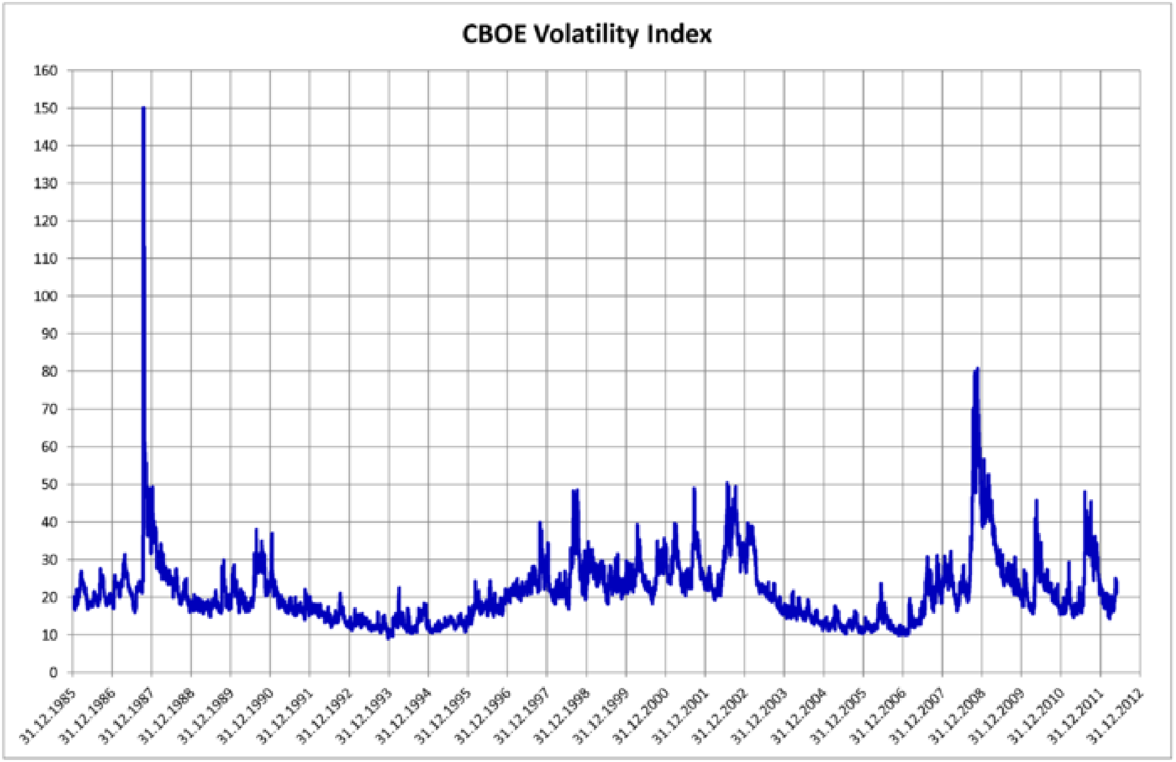
The first observation is that we should be careful in drawing conclusions about *expected values* when the system under consideration may not be ergodic. If it would take extremely large numbers of trajectories to do a reasonable job of exploring the space of possibilities, we will need to find another way of figuring out the typical *time average* of the system.

This also means that building models of economic or financial systems are likely to require thoughtful analysis of any non-ergodic aspects of the system.

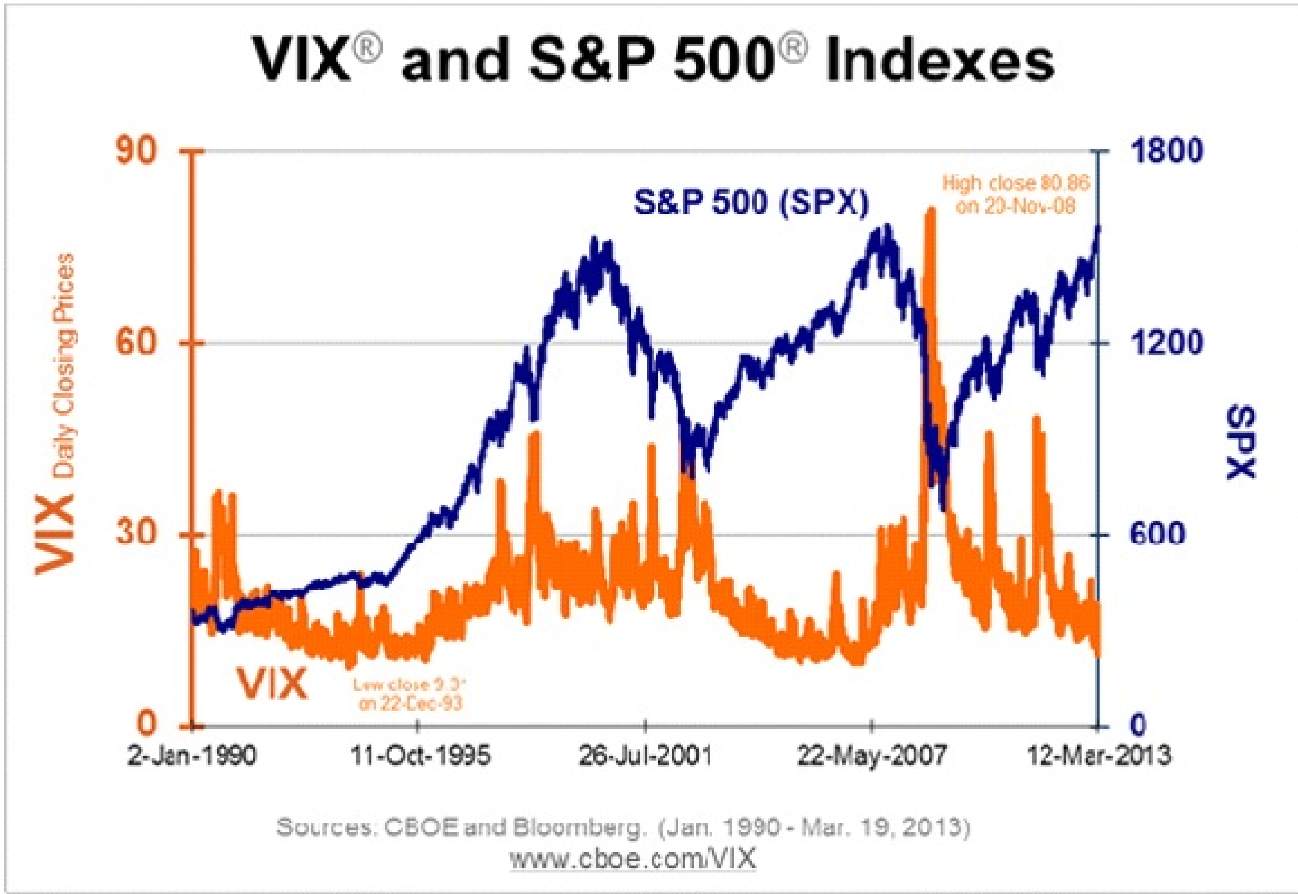
Another observation is that these issues have implications for developing investment portfolios. As we have seen, an investment opportunity with a given mean return (μ) may or may not actually be a good investment, depending on the variance of the system. A system with large variance may not be just riskier, but in fact a bad investment.

This also has consequences for decisions about optimal leverage. If an investor does not understand about the non-ergodic effects of large variance, they may commit excessive leveraged resources to bad investments.

Of course, this is not a simple issue. For example, the standard deviation of an investment opportunity (often called the *volatility*) is likely not to be stationary over time:



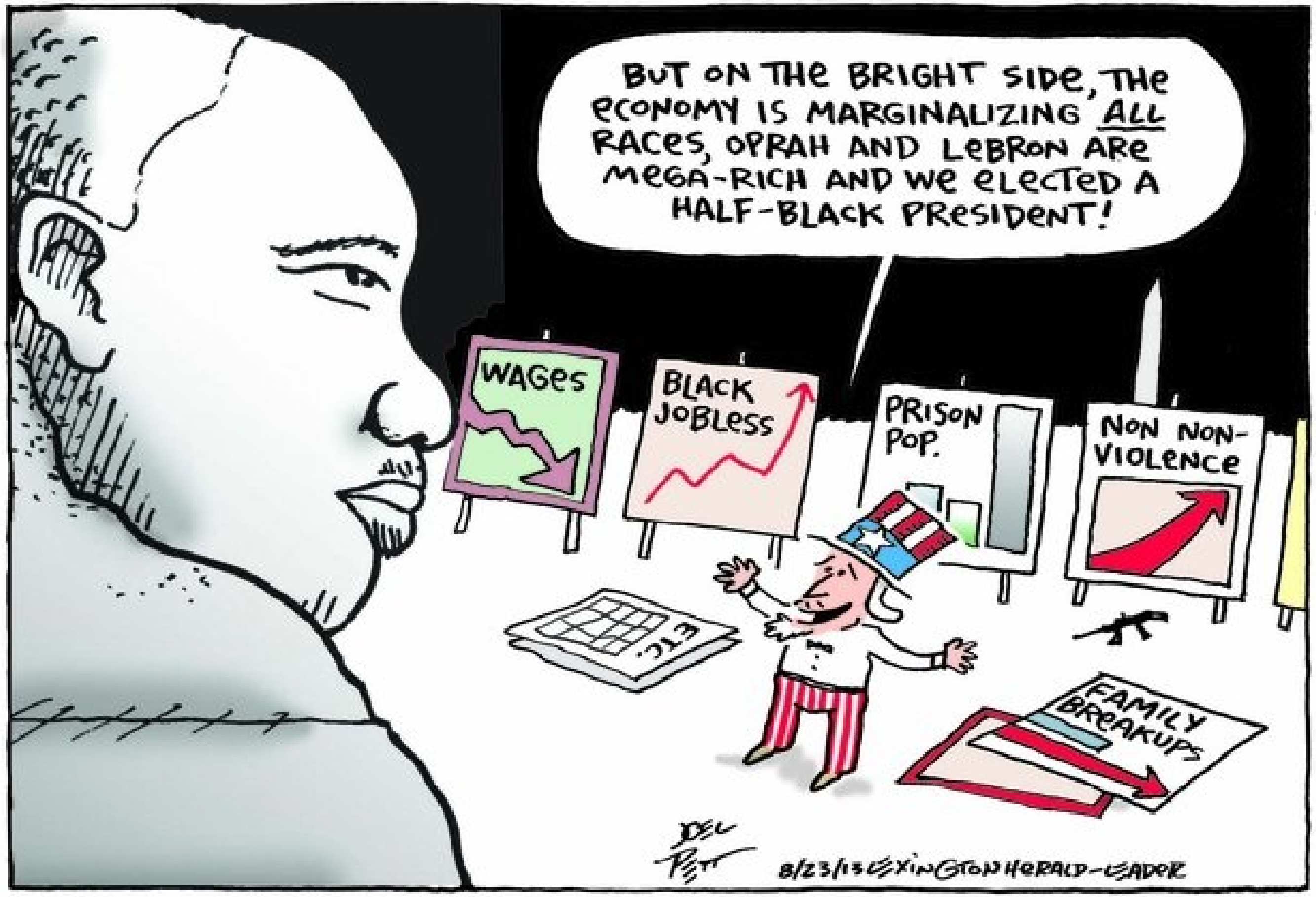
CBOE Volatility Index (VIX) from December 1985 to May 2012 (daily closings), data from Chicago Board Options Exchange



CBOE Volatility Index (VIX) from 1990 to 2013 (daily closings), with S&P 500
data from Chicago Board Options Exchange

In practice, we will be trying to estimate parameters from sampled data, and developing models of various financial instruments. What we have seen here reminds us that we need to understand in some detail both the mean and variance of the instruments.

In addition, a strong determiner of the ensemble average is the extreme values. We need to consider the likelihood of observing those extreme values, and beware of implicitly believing that extreme values are adequately typical to be representative in meaningful ways . . .



Averages . . . cartoon by Joel Pett
http://www.kentucky.com/joel-pett/

The next step, of course, is to extend to both additive and multiplicative random walks, such as the *Kesten Processes*:

$$V(n+1) = V(n) * X(n) + Y(n)$$

where both $X(n)$ and $Y(n)$ are random variables. These processes can lead to *power law* distributions.

And so it goes . . .

Feeling lucky?

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