Nonlinear Systems

(. . . and chaos)

a brief introduction

Tom Carter

Computer Science
CSU Stanislaus

http://csustan.csustan.edu/~ tom/Nonlinear-Systems

September 12, 2011

Our general topics:

What are nonlinear systems? ←

- Let's start by adding another word to this, and ask the question, "What are nonlinear dynamical systems?" (and we'll go back to front . . .)

  A reasonable thing to say is that a *system* is a collection of entities that we can treat (for some purpose, in some context) as a unity of interacting parts or elements. At various times we will treat various collections of entities as systems, or subsystems. We may at times ignore certain elements that might otherwise be included. There will also often be times when we will engage in abstraction, and refer to a "system" when we are actually discussing, manipulating, or analyzing an abstraction from the real (physical) system we are interested in.

A *dynamical* system is one which changes over time. It is generally not unreasonable to assume that there is some form of energy flow involved in such a system.

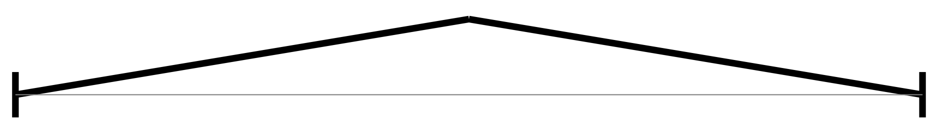
A *nonlinear* dynamical system is, as the name implies, a system whose best description (behavior) is not *linear*. There are various contexts and forms of description for the concept *linear*. We'll look at various examples as we go along.

A Linear Example (string)

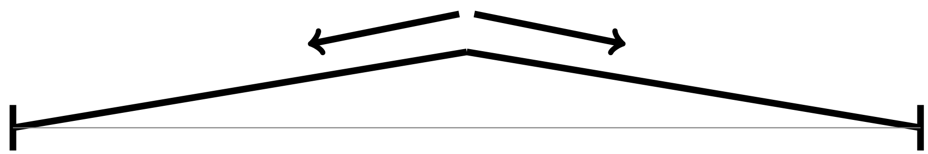
(a little string theory :-)

- It will be worth our while to have some very specific examples available, so let's start with this one – consider a string stretched tightly between two fixed endpoints:

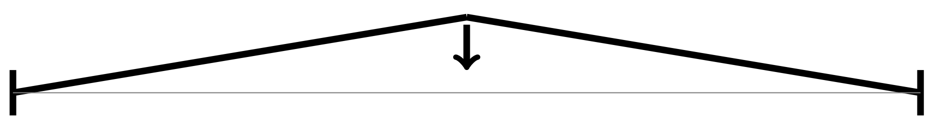
We can pull the center of the string up:



At the moment we release it, there will be forces acting on the string. Let's focus our attention on the center point of the string (where we had taken hold of it to pull it up). There will be forces pulling toward the two fixed ends:

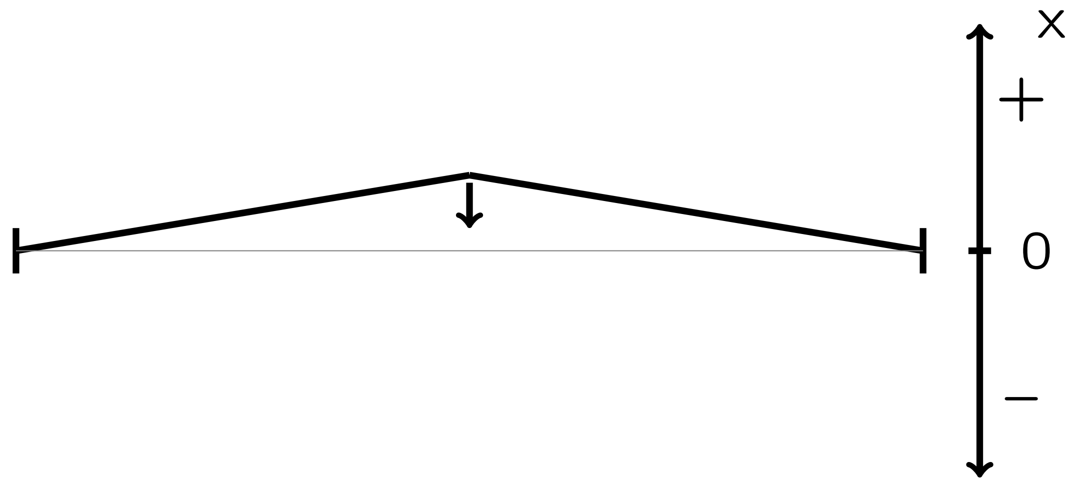


There will be a "net" force acting on the center point of the string:



Now let's put a "reference frame" on the system, so we can make things more explicit.

We will measure the vertical displacement of the center point of the string relative to the "resting position," with positive up and negative down:



At this point we'll make a set of simplifying assumptions, and in particular treat this as a *linear* system (in a sense to be made more explicit below).

- Among the assumptions we'll make are:
  - Everything is nicely symmetric (in particular, the force is exactly vertical).
  - The force is very simple (no complications from friction, etc.).
  - The force changes *linearly* with the displacement of the center point of the string.
  - The motion is always continuous and smooth (differentiable), and thus, also,
  - Newtonian mechanics (and all that that entails, including all the machinery of the calculus . . .)

- Now we'll set this up as a (Newtonian) differential system. The (vertical) displacement of the center point of the string will be denoted by x (which we will recall is actually a function of time t, but we will generally simplify the notation as x rather than $x(t)$). We will (often) denote the derivative of a function with respect to time as

$$\dot{x} = \frac{dx(t)}{dt}$$

and the second derivative as

$$\ddot{x} = \frac{d^2x(t)}{dt^2}.$$

As appropriate, we will refer to velocity and acceleration as

$$v = \dot{x}$$

and

$$a = \dot{v} = \ddot{x}.$$

For the time being, we will assume a very simple form for the force on the center point:

$$F = -x$$

(Note: these should really be vectors $\vec{F} = -\vec{x}$, but we'll keep the notation simple for now . . .). This is the place at which we are doing a *linear* approximation, and thus are working with a *linear dynamical system*.

Now we invoke Newton, and his fundamental equation of motion:

$$F = ma$$

(and we'll use units where $m = 1$, and thus write $F = a$).

- Putting the pieces together, our system is given by the linear ordinary differential equation:

$$a = F = -x$$

or

$$\ddot{x} = -x$$

or

$$\ddot{x} + x = 0.$$

We can use the standard machinery of calculus to solve this differential equation.

The characteristic polynomial of this differential equation is

$$z^2 + 1 = (z + i)(z - i)$$

with roots $z = i$ and $z = -i$. Hence the general form of the solution is

$$x = x(t) = b_0 e^{it} + b_1 e^{-it}.$$

Now we can remember (use) the definition

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

from which we get

$$\begin{aligned}
e^{it} &= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \\
&= 1 + it - \frac{t^2}{2!} - i\frac{t^3}{3!} + \frac{t^4}{4!} + \cdots \\
&= 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots \\
&\qquad + it - i\frac{t^3}{3!} + i\frac{t^5}{5!} - \cdots \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{t^{(2n)}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{t^{(2n+1)}}{(2n+1)!} \\
&= \cos(t) + i \sin(t).
\end{aligned}$$

Then, using some standard techniques (and observing that $\cos(t)$ and $\sin(t)$ are linearly independent), we find that the general solution to our system can be written as:

$$x(t) = c_0 \cos(t) + c_1 \sin(t).$$

Of course, if we didn't want to be reminded of the general method of solving linear ordinary differential equations, or the definition of the exponential function, or Euler's formula ($e^{it} = \cos(t) + i \sin(t)$), we could have just observed that

$$\frac{d^2 \cos(t)}{dt^2} = -\cos(t)$$

and

$$\frac{d^2 \sin(t)}{dt^2} = -\sin(t)$$

and gone straight to our general solution :-)

- There is another nice way to represent such a second order ordinary differential equation, as a system of first order differential equations.

  In this example, we start with the equation

  $$\ddot{x} + x = 0,$$

  then introduce a new variable $v = \dot{x}$ (which we have seen before – the velocity), and then express things as a system:

  $$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -x \end{aligned}$$

  This is a very general approach, which can also be applied to nth order equations, resulting in a system n first order equations.

We can then rewrite the system in vector/matrix notation:

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

(note the convenient use of column notation for vectors).

Changing notation slightly, to simply extension to nth order:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and, writing $\mathbf{x}$ for $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\dot{\mathbf{x}}$ for $\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix}$, and $\mathbf{A}$ for $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, we can write

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}.$$

Generalizing to nth order, writing $\mathbf{x}$ for $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, etc., and also allowing a constant b_i to be added to each row, we have a system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}.$$

In this form, it is easy to see that these are *linear* systems. There is a well developed theory for solving such systems (using eigenvalues/eigenvectors, etc.), and even for generalizations where $\mathbf{A}$ and $\mathbf{b}$ can be time dependent:

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{b}(t).$$

Discussion of this is available in various places, such as here: http://www.unf.edu/~ mzhan/chapter4.pdf.

- For future reference, here is an example of what a *nonlinear* system would like in this sort of notation (of course, the right hand sides won't be linear, and hence we won't get the nice simple matrix representation).

  These are the (famous) *Lorenz equations*, which give rise to the *Lorenz attractor*:

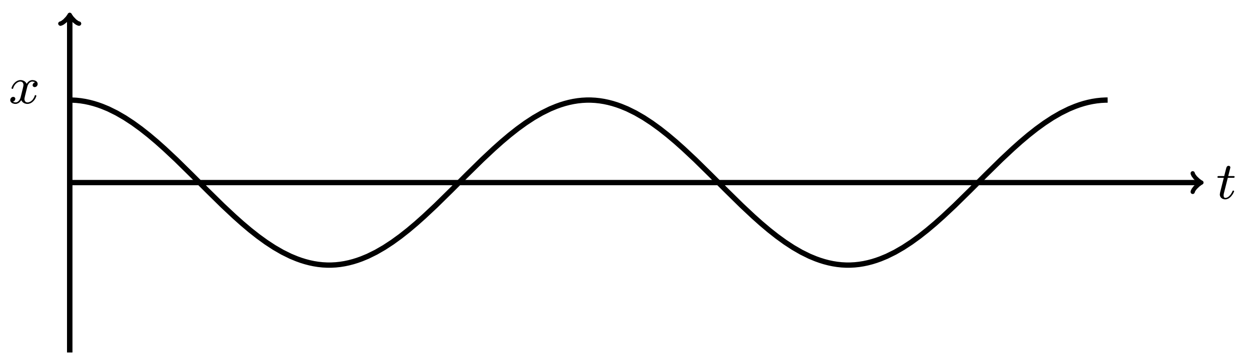
$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = x(\rho - z) - y$$
$$\dot{z} = xy - \beta z$$

  where σ is the *Prandtl number*, ρ is the *Rayleigh number*, and β is another adjustable parameter. All of σ, ρ, and β are positive, with typical values $\sigma = 10$, $\beta = 8/3$, and ρ is varied. When $\rho = 28$, the system exhibits chaotic behavior. You can see the nonlinear (mixed) terms in the second and third equations (the xz and xy terms).

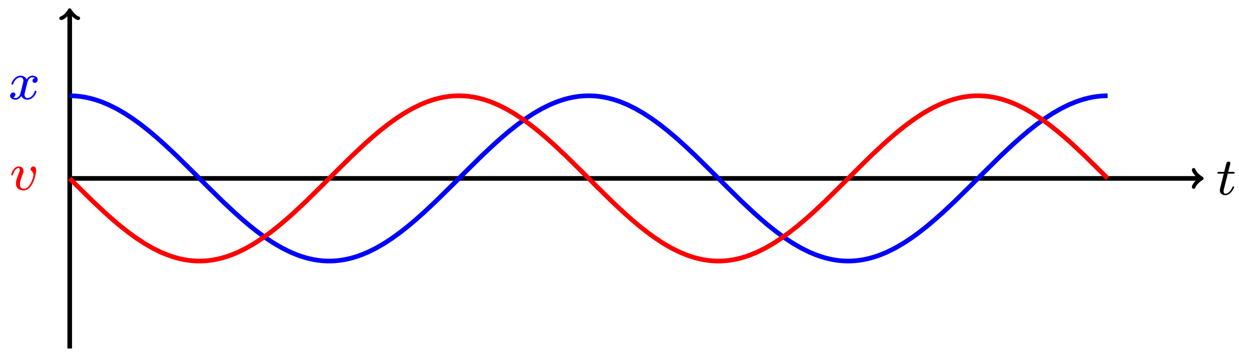
- Let's go back to our string example (i.e., our simple harmonic oscillator). For ease of reading, let's also go back to our notation of position x and velocity v, or, when we want to clarify the time dependence, $x(t)$ and $v(t)$:

$$\dot{x} = v \qquad\qquad \dot{x}(t) = v(t)$$
$$\dot{v} = -x \qquad\qquad \dot{v}(t) = -x(t)$$

The simplest solution to this system is $x(t) = \cos(t)$ (and, of course, $v(t) = -\sin(t)$). We can visualize the behavior of the system by plotting $x(t)$:



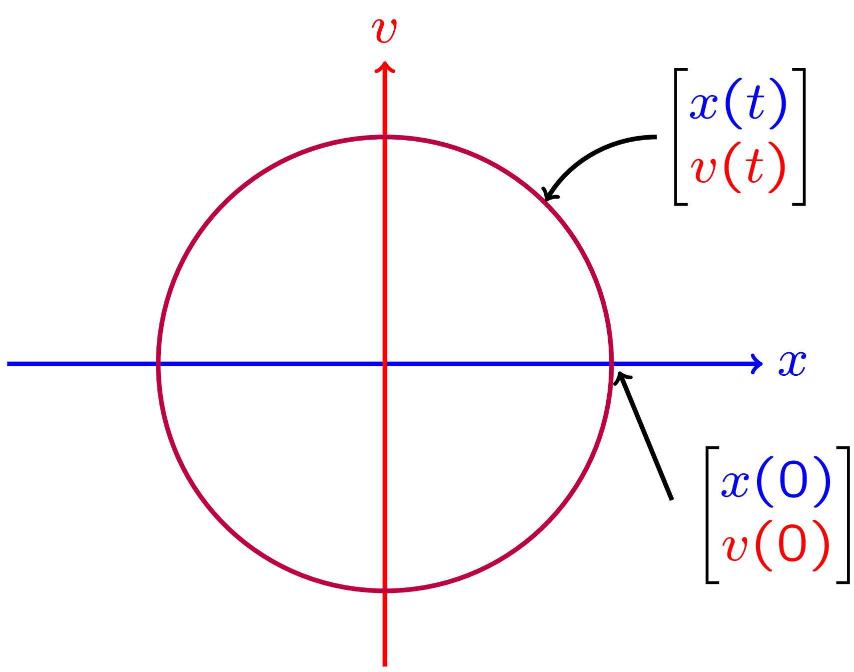
We can also try plotting both $x(t)$ and $v(t)$ on the same graph:



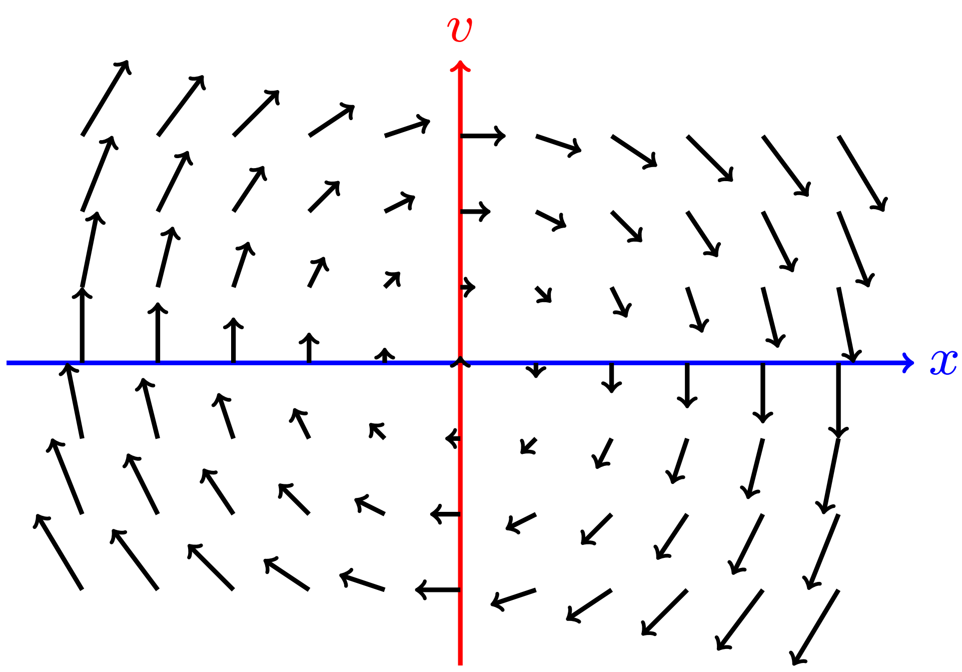
Before long, this can get fairly messy. We would like better ways to visualize the system.

One important point is that this is a fully deterministic system, and the state of the system is completely specified once we know x and v. We can thus visualize the system in *state space* (also called *phase space*), with one dimension for each variable. Each point in phase space represents the state of the system at a particular time, and over time the system will trace a *trajectory* through phase space.

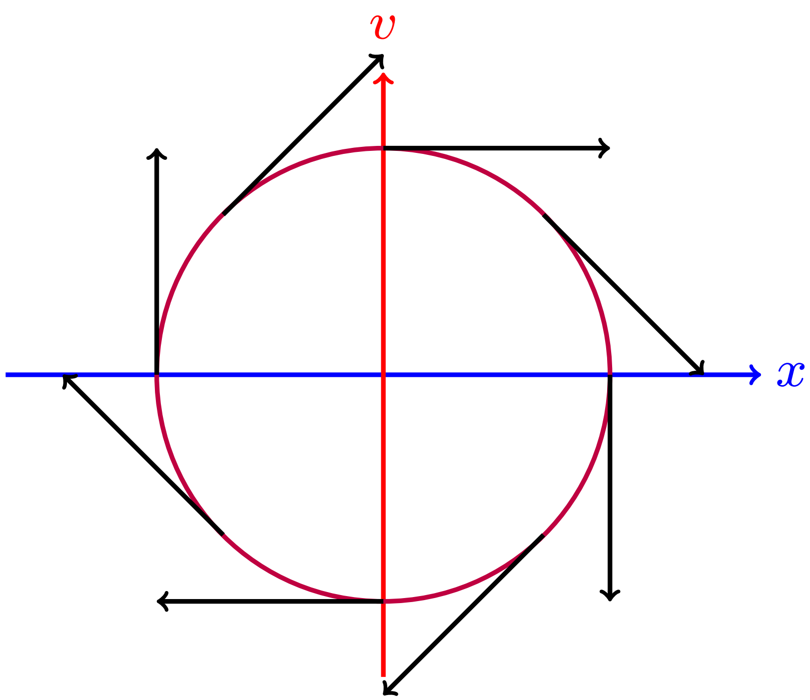
For our simple harmonic oscillator, the phase space trajectory takes a particularly nice form:



- We can also think about the vector field consisting of the derivatives. Associated with each point in phase space, there is the vector of the velocity/acceleration values. At the point $\begin{bmatrix} x(t) \\ v(t) \end{bmatrix}$ there is the derivative vector $\begin{bmatrix} v(t) \\ -x(t) \end{bmatrix}$:



For a given trajectory in phase space, the derivative vectors along the trajectory will be tangent to the trajectory at each point:



We can even use this to find (at least an approximation to) a trajectory, given an initial point (initial condition) in phase space. The basic idea comes from the definition of the derivative:

$$\frac{df(t)}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

which means that, for small Δt, we have the approximation

$$\frac{df(t)}{dt} \approx \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

or,

$$\frac{df(t)}{dt} * \Delta t \approx f(t + \Delta t) - f(t)$$

and thus the approximation

$$f(t + \Delta t) \approx f(t) + \frac{df(t)}{dt} * \Delta t.$$

In the case of our simple harmonic oscillator, we have the approximation

$$\begin{bmatrix} x(t+\Delta t) \\ v(t+\Delta t) \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \dot{x}(t) * \Delta t \\ \dot{v}(t) * \Delta t \end{bmatrix}$$
$$= \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} v(t) * \Delta t \\ -x(t) * \Delta t \end{bmatrix}.$$

Working with our particular system, let's do a couple of steps, starting at $t = 0$, and with $\Delta t = 0.1$. We will have $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} \dot{x}(0) \\ \dot{v}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. Our first step of approximation will give us

$$\begin{bmatrix} x(0.1) \\ v(0.1) \end{bmatrix} \approx \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 * 0.1 \\ -1 * 0.1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ -0.1 \end{bmatrix}.$$

The next three steps of our approximation will be:

$$\begin{bmatrix} x(0.2) \\ v(0.2) \end{bmatrix} \approx \begin{bmatrix} 1 \\ -0.1 \end{bmatrix} + \begin{bmatrix} -0.1 * 0.1 \\ -1 * 0.1 \end{bmatrix} = \begin{bmatrix} 0.99 \\ -0.2 \end{bmatrix}.$$

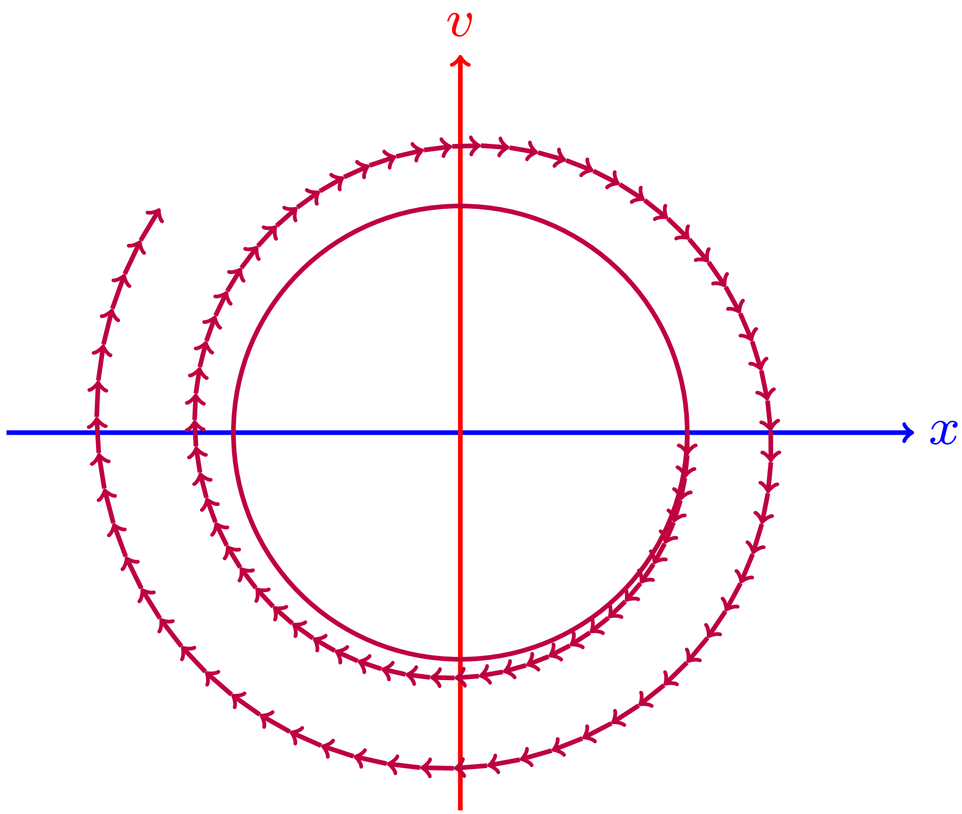
and

$$\begin{bmatrix} x(0.3) \\ v(0.3) \end{bmatrix} \approx \begin{bmatrix} 0.99 \\ -0.2 \end{bmatrix} + \begin{bmatrix} -0.2 * 0.1 \\ -0.99 * 0.1 \end{bmatrix} = \begin{bmatrix} 0.97 \\ -0.299 \end{bmatrix}$$

and

$$\begin{bmatrix} x(0.4) \\ v(0.4) \end{bmatrix} \approx \begin{bmatrix} 0.97 \\ -0.299 \end{bmatrix} + \begin{bmatrix} -0.299 * 0.1 \\ -0.97 * 0.1 \end{bmatrix} = \begin{bmatrix} 0.9401 \\ -0.396 \end{bmatrix}$$

This will look like (I have gone ahead and added several more steps):



The approximation starts out doing a reasonable job (but over time it does drift away from the real trajectory . . .).

- We'll keep in mind this approximation approach, and come back to it later. In particular, we'll need to think some about what cautions we should keep in mind when using approximations. But, let's briefly return now to the thrilling days of yesteryear, and do a little bit more with linear systems.

  Our linear approximation of the string system clearly leaves out a bunch of stuff. In particular, we know perfectly well that a vibrating string won't go on vibrating forever. Things like friction (both internal within the string itself, and external, like air resistance) will play a role in the dynamics. We can improve our system by adding a friction term. In keeping with our simplification of linearity, a reasonable approximation of the friction term is that it depends linearly on velocity. Keeping terms and constants simple, we can learn a reasonable amount by studying this system (the second term in the $\dot{v}$ line is friction):

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ -x - bv \end{bmatrix}.$$

Using the "characteristic polynomial" approach, we can look at $\ddot{x} = -x - b\dot{x}$ or $\ddot{x} + b\dot{x} + x = 0$, and thus work with

$$r^2 + br + 1 = 0$$

giving us

$$r = \frac{-b \pm \sqrt{b^2 - 4}}{2}$$

or, assuming $|b| < 2$,

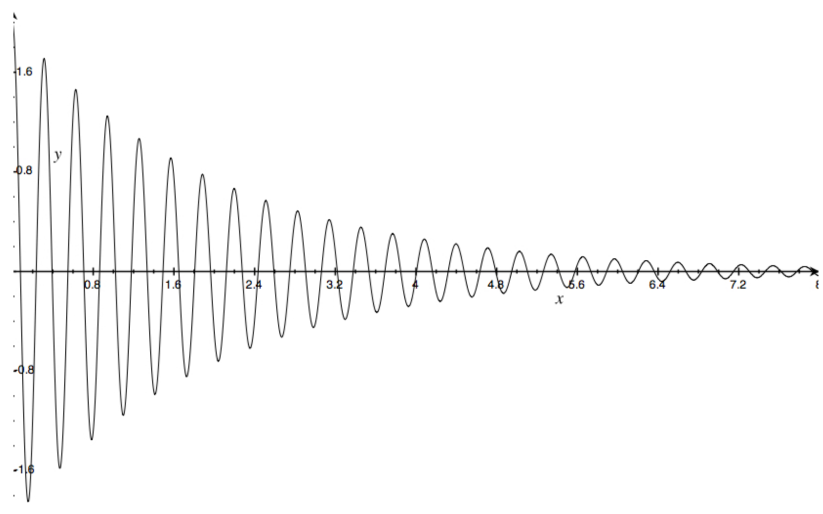
$$r = -\frac{b}{2} \pm i\frac{\sqrt{4 - b^2}}{2}$$

.

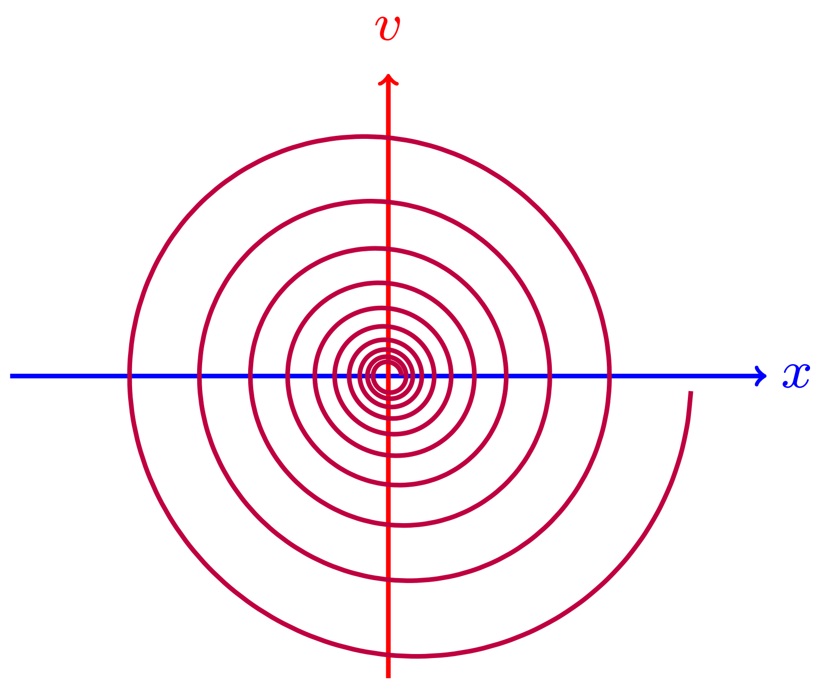
From this, we get a solution to the system in the form

$$
\begin{aligned}
x(t) &= e^{(-\frac{b}{2}+i\frac{\sqrt{4-b^2}}{2})t} + e^{(-\frac{b}{2}-i\frac{\sqrt{4-b^2}}{2})t} \\
&= e^{-\frac{b}{2}t}\left(e^{i\frac{\sqrt{4-b^2}}{2}t} + e^{-i\frac{\sqrt{4-b^2}}{2}t}\right) \\
&= e^{-\frac{b}{2}t} * 2\cos\left(\frac{\sqrt{4-b^2}}{2}t\right)
\end{aligned}
$$

which is decaying oscillations:

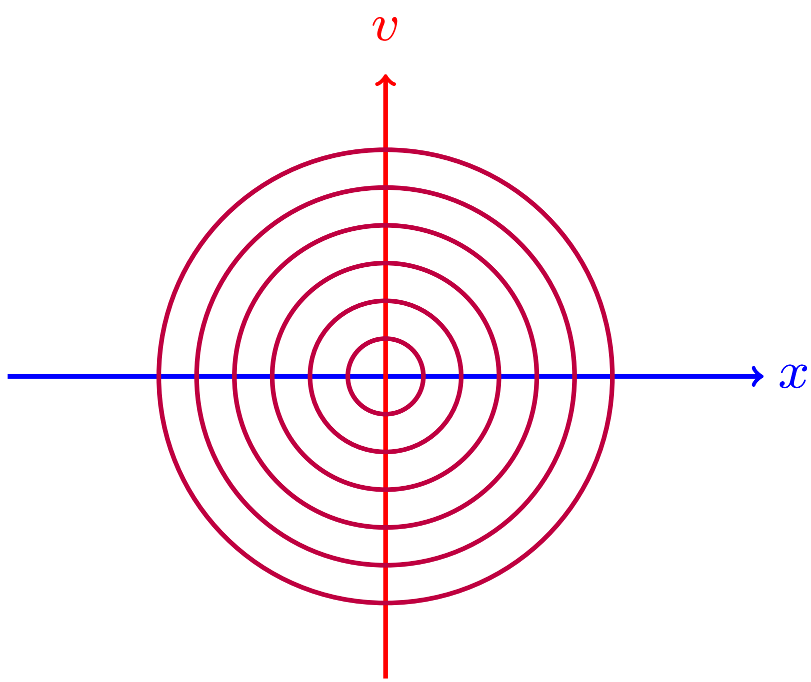


In phase space, this system spirals in toward $(0,0)$ (see appendix 1):



- We can characterize various points in phase space, and various trajectories.

  In the case of the simple harmonic oscillator, for any starting point (initial condition), the resulting trajectory is a simple closed trajectory – a *periodic orbit*.



In the special case of starting at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, the system is at a *fixed point*. This fixed point is, however, not *stable*, in the sense that if some noise jostles the system, it will follow a periodic orbit somewhat away from $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

On the other hand, in the case of of the damped harmonic oscillator, any initial condition will tend toward $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as $t \to \infty$. $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a fixed point, and is an *attracting stable* fixed point, in the sense that if noise jostles the system, it will, as $t \to \infty$, return to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

For any of these continuous systems (i.e., systems where t is a continuous variable), we can get at new system by replacing t with $-t$. In the case of the simple harmonic oscillator, there is complete symmetry, and the new system is indistinguishable from the original – in particular, all trajectories are closed periodic orbits.

In the case of the damped harmonic oscillator, there is a fixed point at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, but all other trajectories spiral out (in phase space) away from $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. In this case, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an *unstable, repelling* fixed point.

Discrete Linear Systems ←

- The systems we have been looking at all have time (t) as a continuous variable (and we have been limiting ourselves to smooth, i.e., differentiable, systems).

  Now I'd like to move to *discrete* systems, where time proceeds in steps rather than continuously. In other words, instead of looking at the variable $x(t)$, we will now take time to only take integral values, and we will be interested in systems with values $x_0, x_1, x_2, \ldots$

  Instead of *differential* equations, we will be working with *difference* equations. The general form of a (one variable) difference equation is

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_0, c).$$

A very simple example is the Fibonacci sequence. This is given by the difference equation

$$x_{n+2} = x_{n+1} + x_n.$$

This is a second order difference equation. Once we specify initial conditions (e.g., $x_0 = 0$, $x_1 = 1$), we have the solution sequence $0, 1, 1, 2, 3, 5, 8, 13, 21, \dots$

We can solve linear difference equations for the general solution for the nth term using a procedure similar to the approach for linear differential equations. For the Fibonacci sequence, we rewrite the equation as

$$x_{n+2} - x_{n+1} - x_n = 0,$$

and then work with the characteristic polynomial

$$r^2 - r - 1 = 0.$$

From this, we get

$$r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

The general form of the solution is then

$$x_n = a_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + a_2 \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Once we specify the initial conditions, we can solve for a_1 and a_2. For example, when $x_0 = 1$ and $x_1 = 1$, we get

$$a_1 * 1 + a_2 * 1 = 0$$

$$a_1 * \left(\frac{1+\sqrt{5}}{2}\right) + a_2 * \left(\frac{1-\sqrt{5}}{2}\right) = 1,$$

from which $a_2 = -a_1$, and so

$$a_1 * \left(\frac{1+\sqrt{5}}{2}\right) - a_1 * \left(\frac{1-\sqrt{5}}{2}\right) = 1,$$

and hence

$$a_1 = \frac{1}{\sqrt{5}}$$

$$a_2 = -\frac{1}{\sqrt{5}}$$

and so

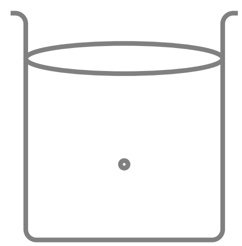
$$x_n = \frac{1}{\sqrt{5}} * \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} * \left(\frac{1-\sqrt{5}}{2}\right)^n .$$

This is a very general approach (with some minor subtleties when dealing with repeated roots) – and, of course, there are the problems of finding the roots for higher order polynomials, but this basically gets us going on general linear difference equations.

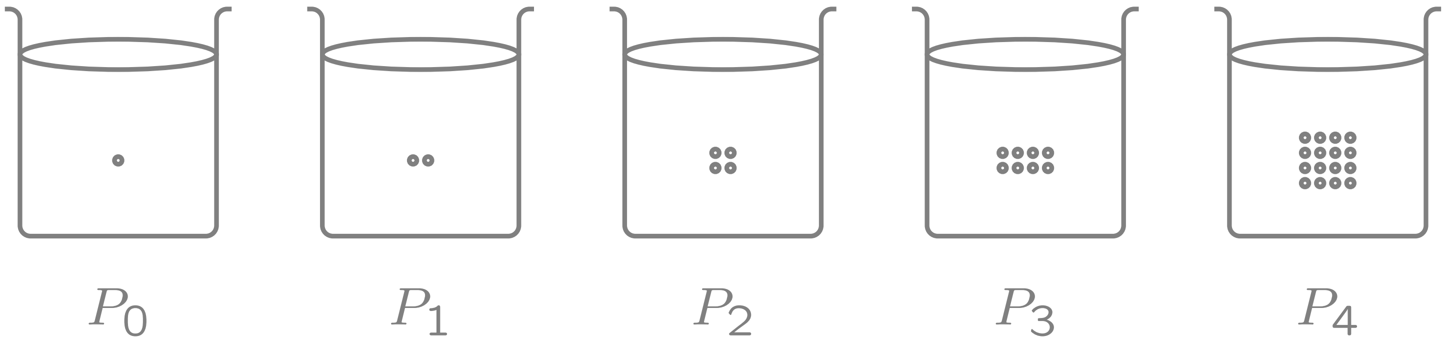
The Logistics Equation (derivation) ←

- Now it's time to get down to business, and start exploring a specific nonlinear example. We're going to look at a classic example from biology, concerning population growth.

  We'll start by considering a single species, living in an environment where it depends on a consumable resource. So, imagine a beaker of sugar water, into which we put a single bacterium:



As time passes, the bacteria reproduce (by dividing, time lapse pictures, bacteria greatly magnified . . .):



The population of bacteria at time $n + 1$ will be given by

$$P_{n+1} = 2P_n \quad \text{(with } P_0 = 1\text{)},$$

and so in general

$$P_n = 2^n.$$

- Now we'll ask the traditional question at this point: If we put the first bacterium in the beaker at 9:00 am, the bacteria divide every 10 minutes, and the beaker is completely full of bacteria at 12:00 noon, at what time was the beaker exactly half full of bacteria?

  The answer is clearly "11:50 am." (Now we're supposed to think about natural resources like oil, and oil consumption doubling every 20 years – if in all the time of oil consumption up until now we have consumed half the total reserve of oil, in how many years will we have consumed all the oil? etc. . . .)

  But the question I really want to ask is, "What will the beaker look like at 10 minutes after 12:00 noon?"

  The first thought is that there will be sugar water and bacteria all over the table because the beaker will have overflowed from the next doubling (it was completely full at 12:00 noon) – but in the real world, that can't be right. In fact, there won't be any live bacteria in the beaker at 10 minutes after 12:00 noon, because they all will have died of starvation! The beaker was completely full of bacteria, so there was no sugar water left . . .

In fact, as the bacteria population gets closer to filling the beaker, there will be a downward pressure on population – there will be increasing competition for scarce resources. Therefore, a better model of the bacteria population at a given time $n + 1$ will be

$$P_{n+1} = 2x_n(M - P_n)$$

where the 2 is from population growth by doubling, and M is the maximum number of bacteria that can fit in the beaker. Note that this corresponds with the idea that if the beaker is ever completely full of bacteria, in the next time step the population will go to 0 because there will be mass starvation . . .

- Now let's simplify the units – instead of keeping track of the total population P_n, we'll let $x_n = \frac{P_n}{M}$, that is, the proportion of the maximum population (also sometimes called the *carrying capacity*) that we have at time n. We will thus have $0 \leq x_n \leq 1$. Let's also generalize to other species that might have a net birth rate R other than 2, so that we will have the classic *logistics equation* for population of a single species in a resource limited environment:

$$x_{n+1} = R * x_n * (1 - x_n).$$

Note that this has an x_n^2 term, is thus nonlinear, and can't be "solved" in any straightforward way . . .

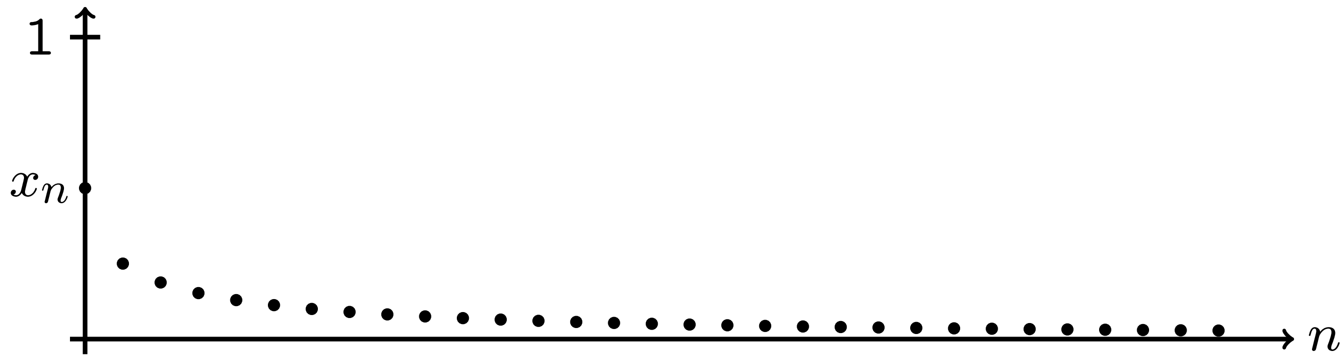
The Logistics Equation (analysis) $\leftarrow$

- The fact that we can't "solve" the logistics equation doesn't mean that we can't study its behavior, or analyze its characteristics. Let's do some work on that. We can start by just observing the system in action.

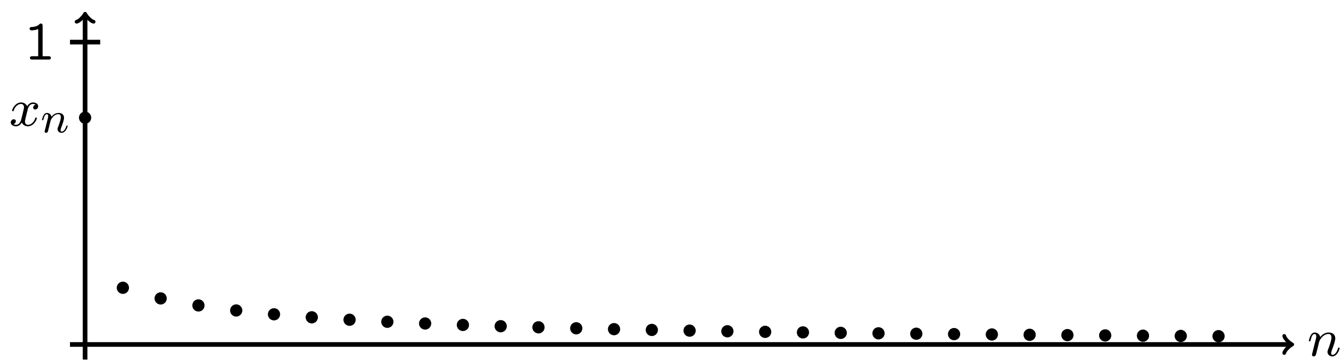
  For our first example, let's look at what happens when $R = 1$. We'll choose a starting value of $x_0 = \frac{1}{2}$. We will have

$$
\begin{aligned}
x_0 &= \frac{1}{2} \\
x_1 &= 1 * \frac{1}{2} * (1 - \frac{1}{2}) = \frac{1}{4} \\
x_2 &= 1 * \frac{1}{4}(1 - \frac{1}{4}) = \frac{1}{4} * \frac{3}{4} = \frac{3}{16} \\
&\ldots
\end{aligned}
$$

Plotting x_n vs. n, we see (first $x_0 = \frac{1}{2}$, then $x_0 = 0.75$):

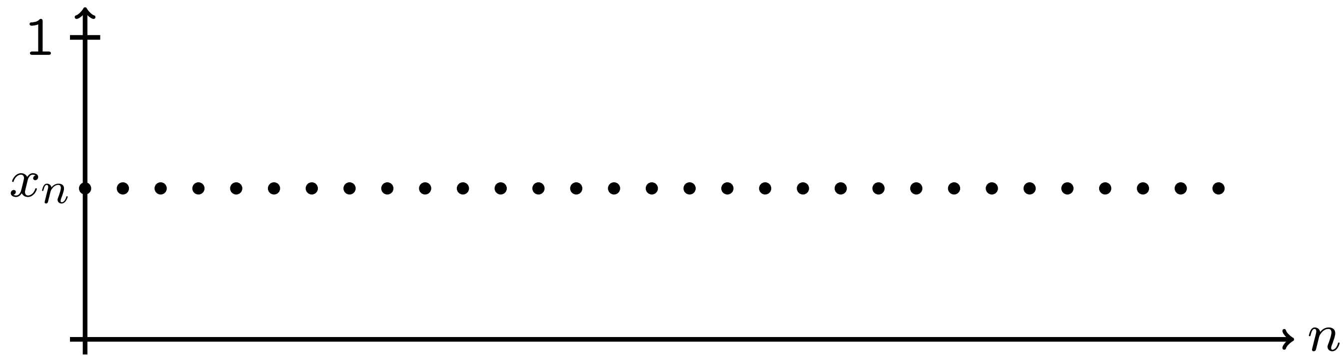


$$x_{n+1} = 1.0 * x_n * (1 - x_n), \quad x_0 = \frac{1}{2}$$

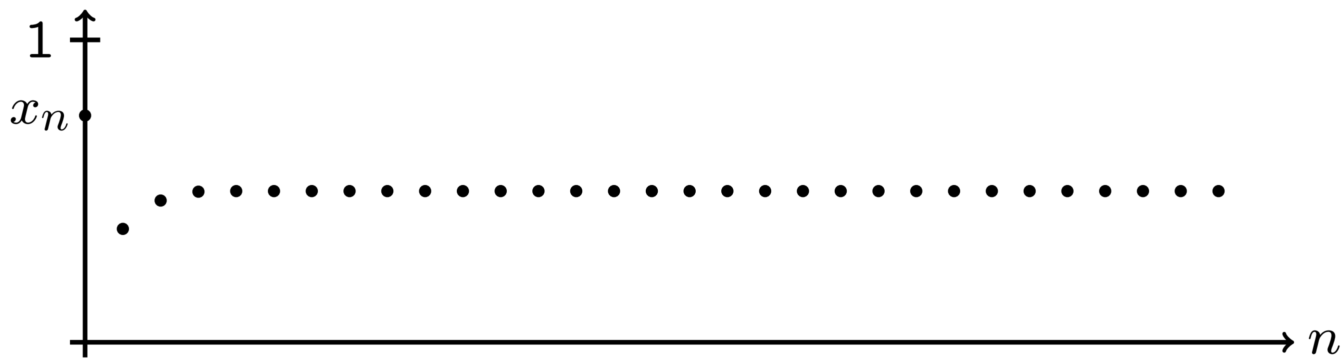


$$x_{n+1} = 1.0 * x_n * (1 - x_n), \quad x_0 = 0.75$$

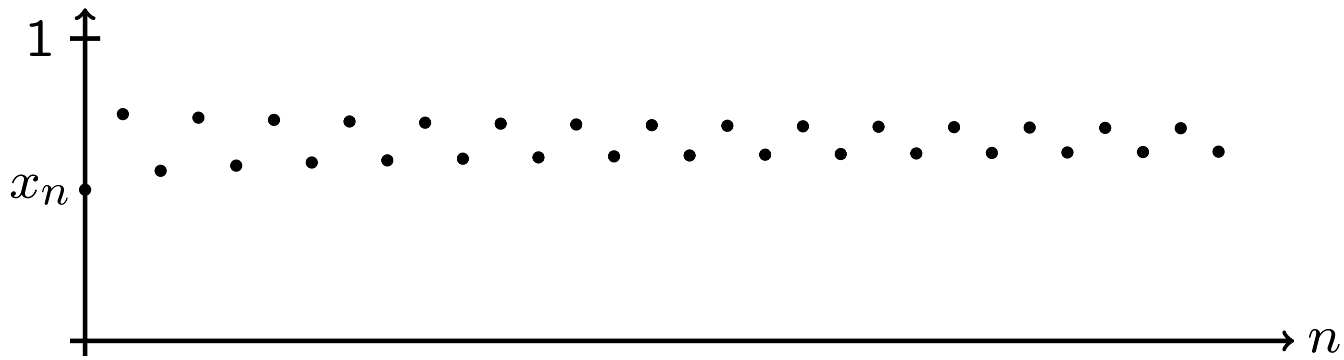
Let's look at some other values of R (and values of x_0):



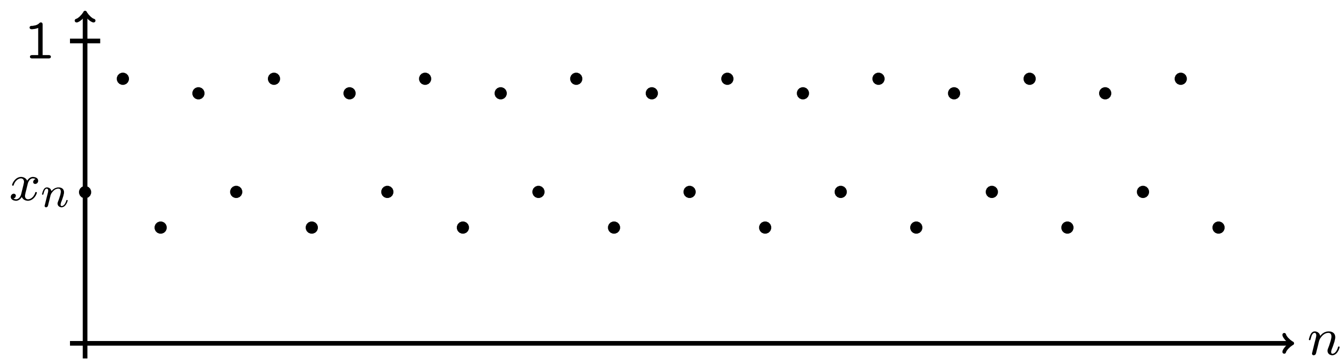
$$x_{n+1} = 2.0 * x_n * (1 - x_n), \quad x_0 = 0.5$$



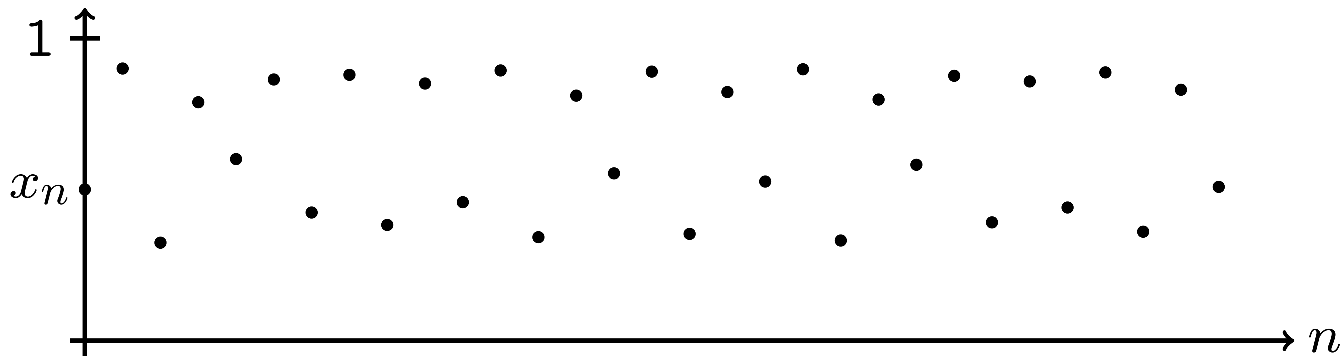
$$x_{n+1} = 2.0 * x_n * (1 - x_n), \quad x_0 = 0.75$$



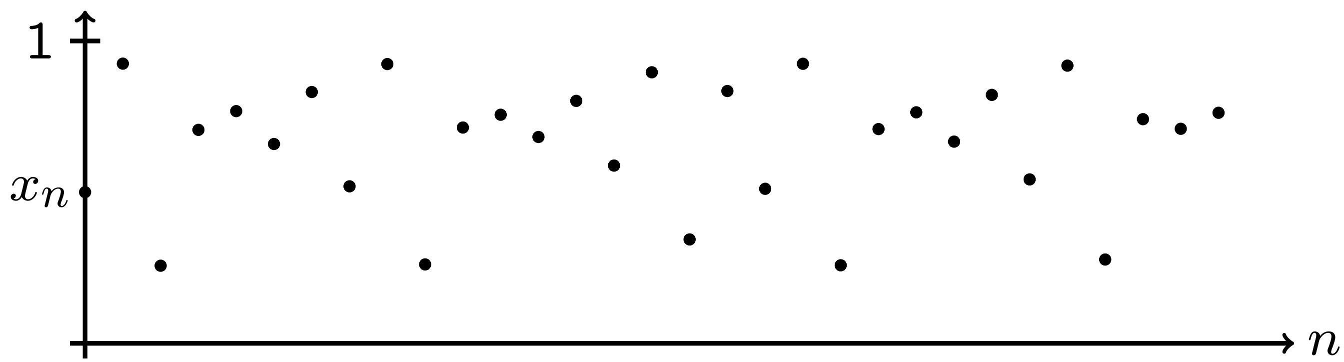
$$x_{n+1} = 3.0 * x_n * (1 - x_n), \quad x_0 = 0.5$$



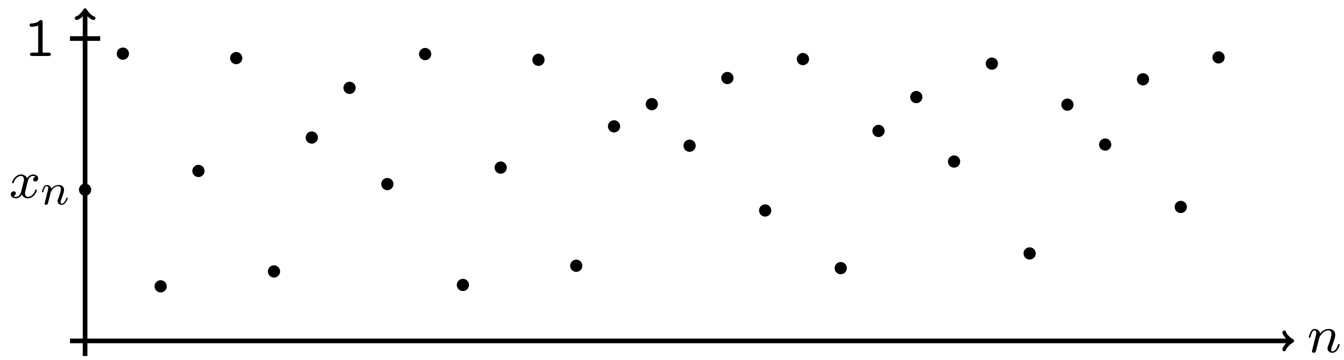
$$x_{n+1} = 3.5 * x_n * (1 - x_n), \quad x_0 = 0.5$$



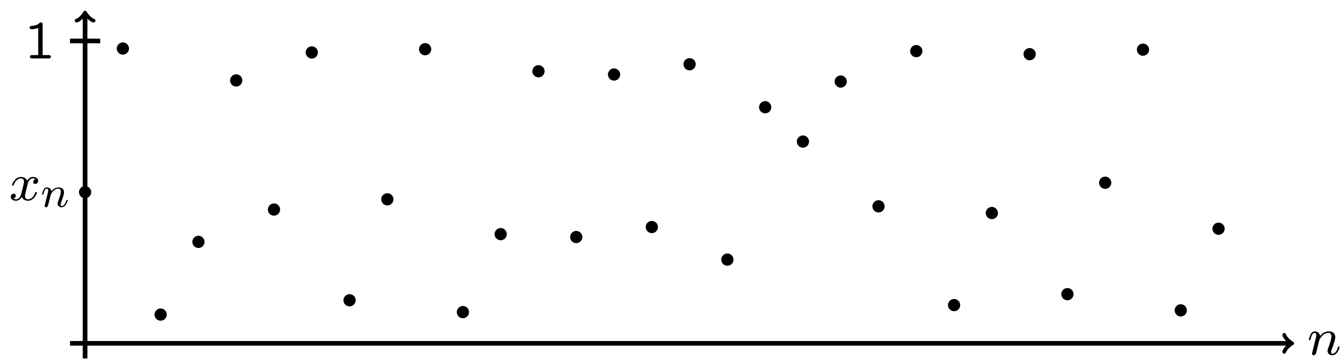
$$x_{n+1} = 3.6 * x_n * (1 - x_n), \quad x_0 = 0.5$$



$$x_{n+1} = 3.7 * x_n * (1 - x_n), \quad x_0 = 0.5$$

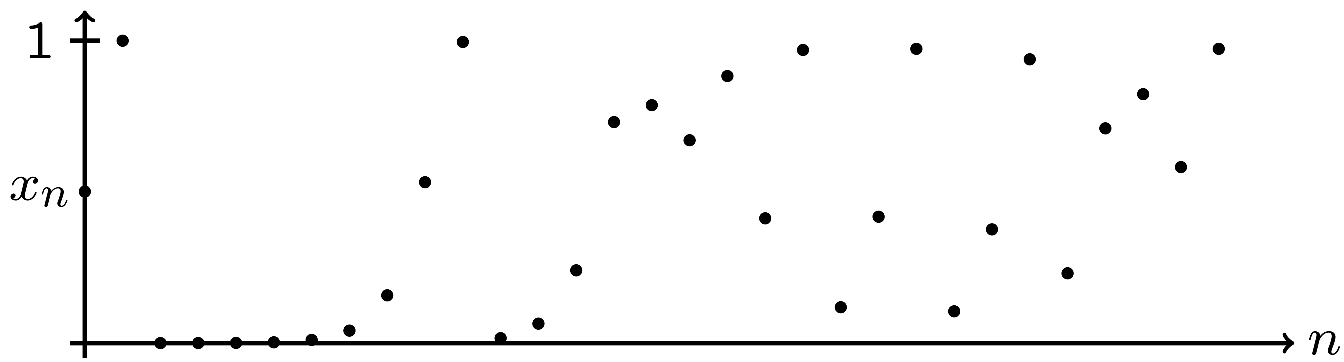


$$x_{n+1} = 3.8 * x_n * (1 - x_n), \quad x_0 = 0.5$$

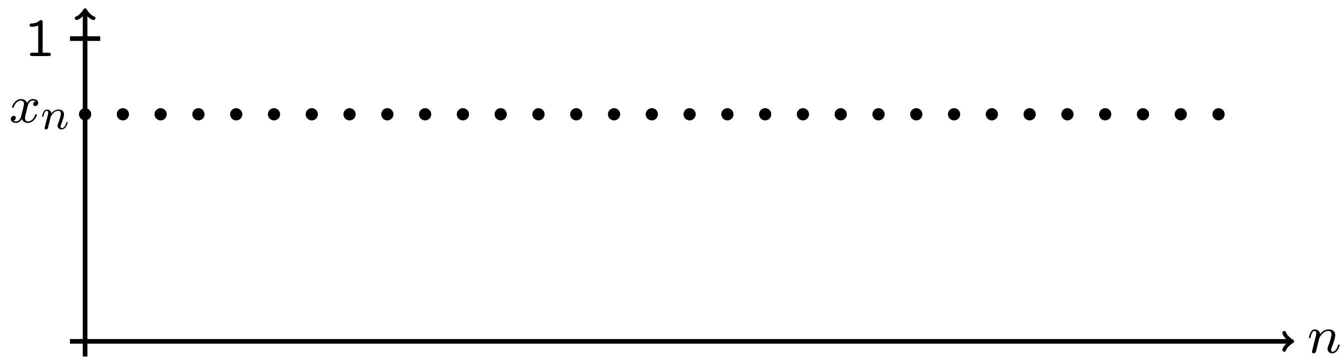


$$x_{n+1} = 3.9 * x_n * (1 - x_n), \quad x_0 = 0.5$$

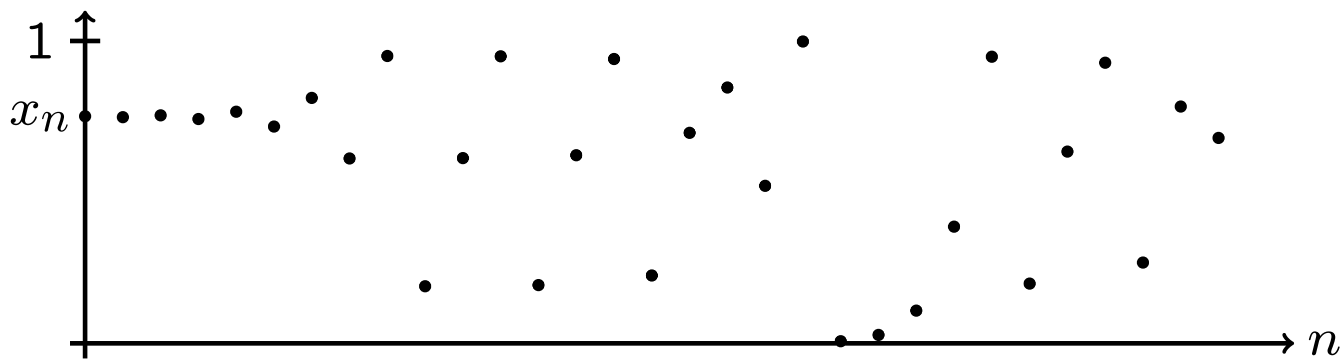
$$x_{n+1} = 4.0 * x_n * (1 - x_n), \quad x_0 = 0.5$$



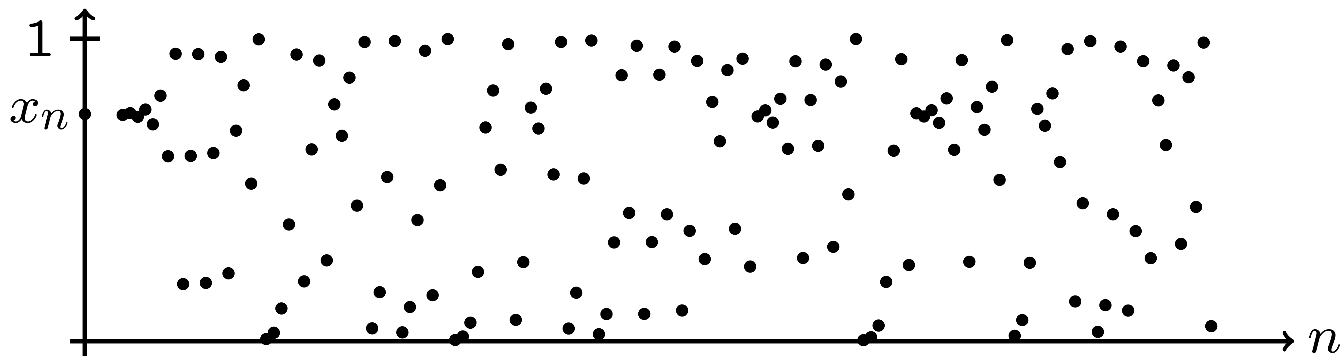
$$x_{n+1} = 4.0 * x_n * (1 - x_n), \quad x_0 = 0.501$$



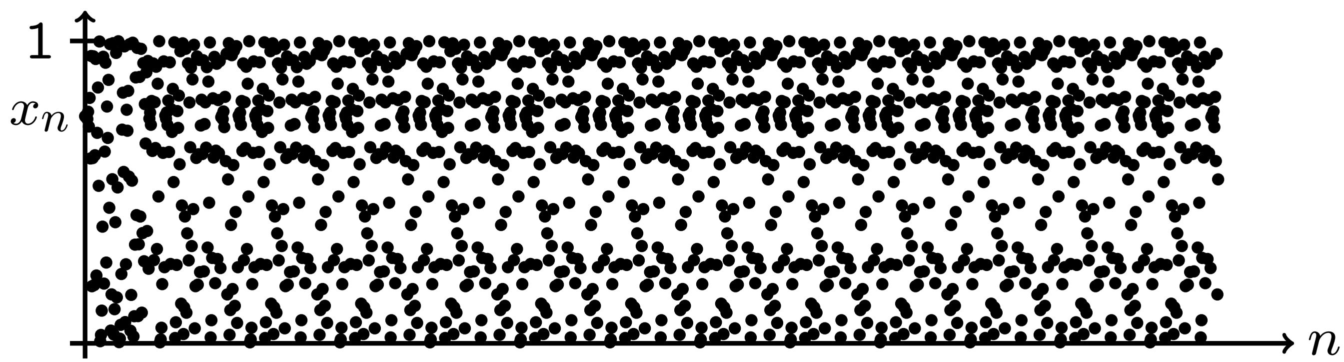
$$x_{n+1} = 4.0 * x_n * (1 - x_n), \quad x_0 = 0.75$$



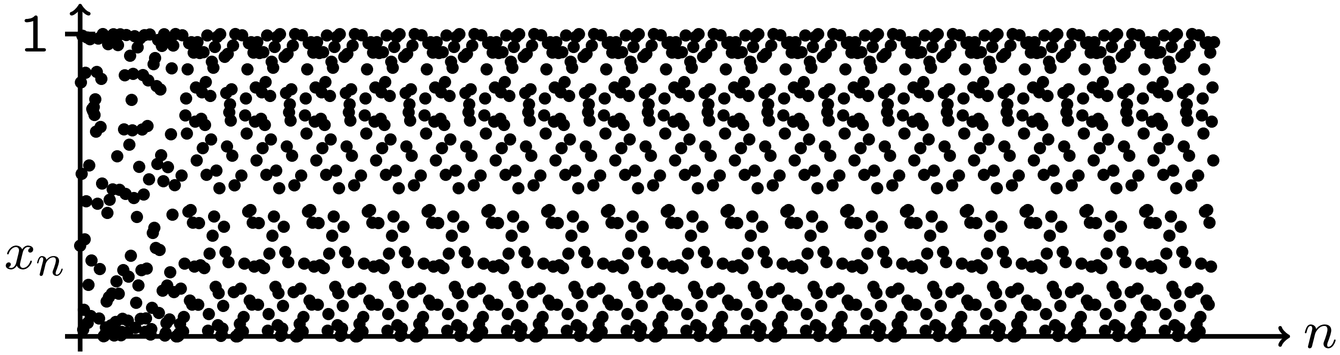
$$x_{n+1} = 4.0 * x_n * (1 - x_n), \quad x_0 = 0.751$$



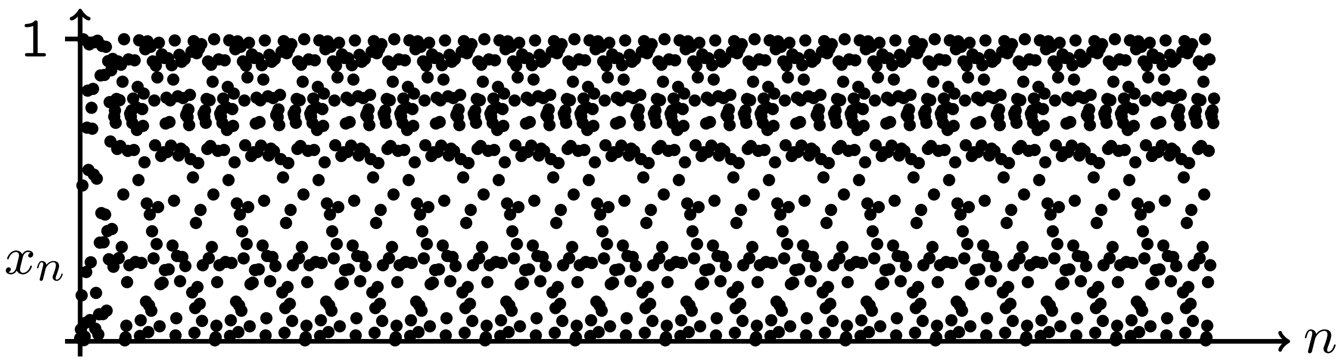
$x_{n+1} = 4.0 * x_n * (1 - x_n), \quad x_0 = 0.751$



$x_{n+1} = 4.0 * x_n * (1 - x_n), \quad x_0 = 0.751$

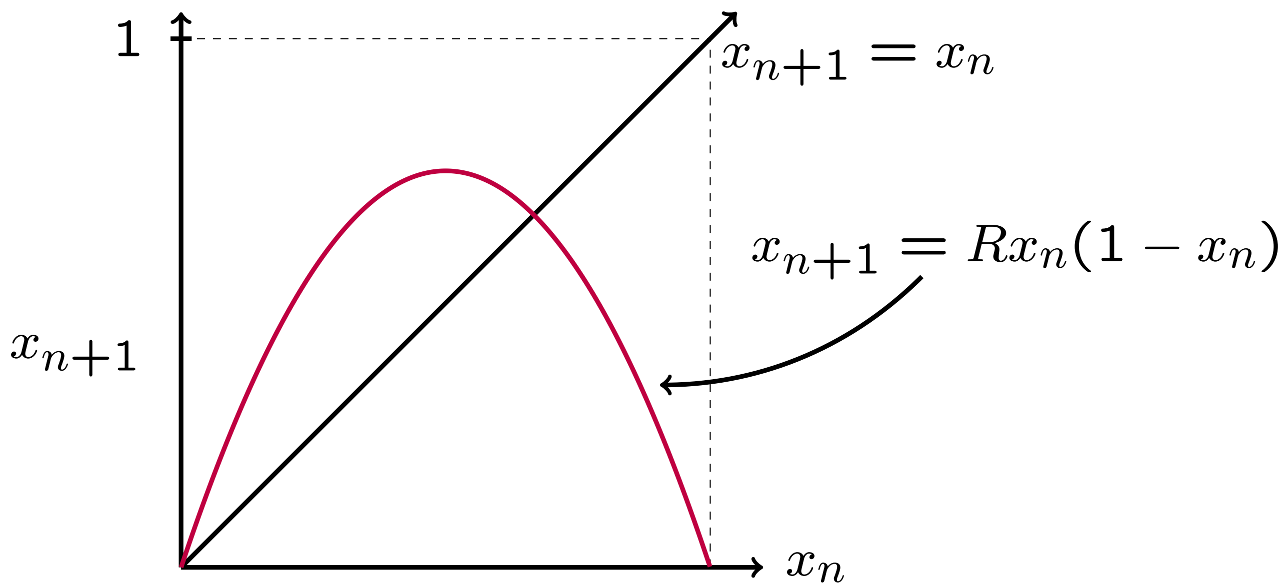


$$x_{n+1} = 4.0 * x_n * (1 - x_n), \quad x_0 = 0.3$$

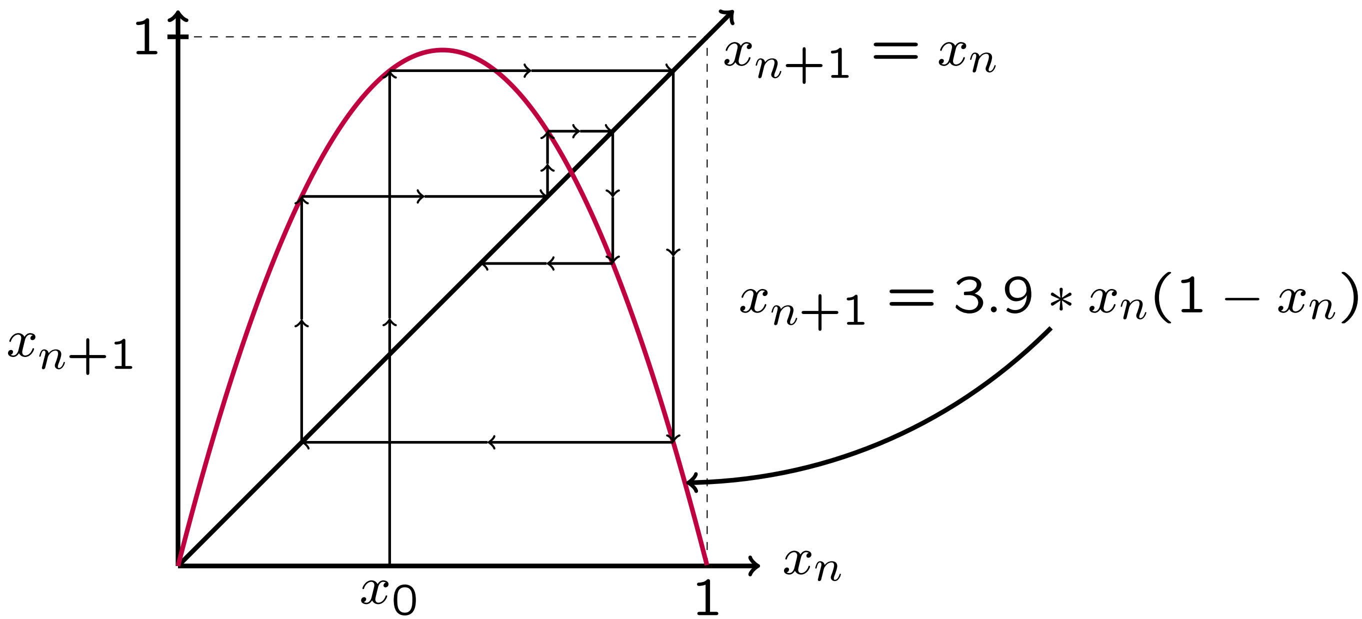


$$x_{n+1} = 4.0 * x_n * (1 - x_n), \quad x_0 = 0.01$$

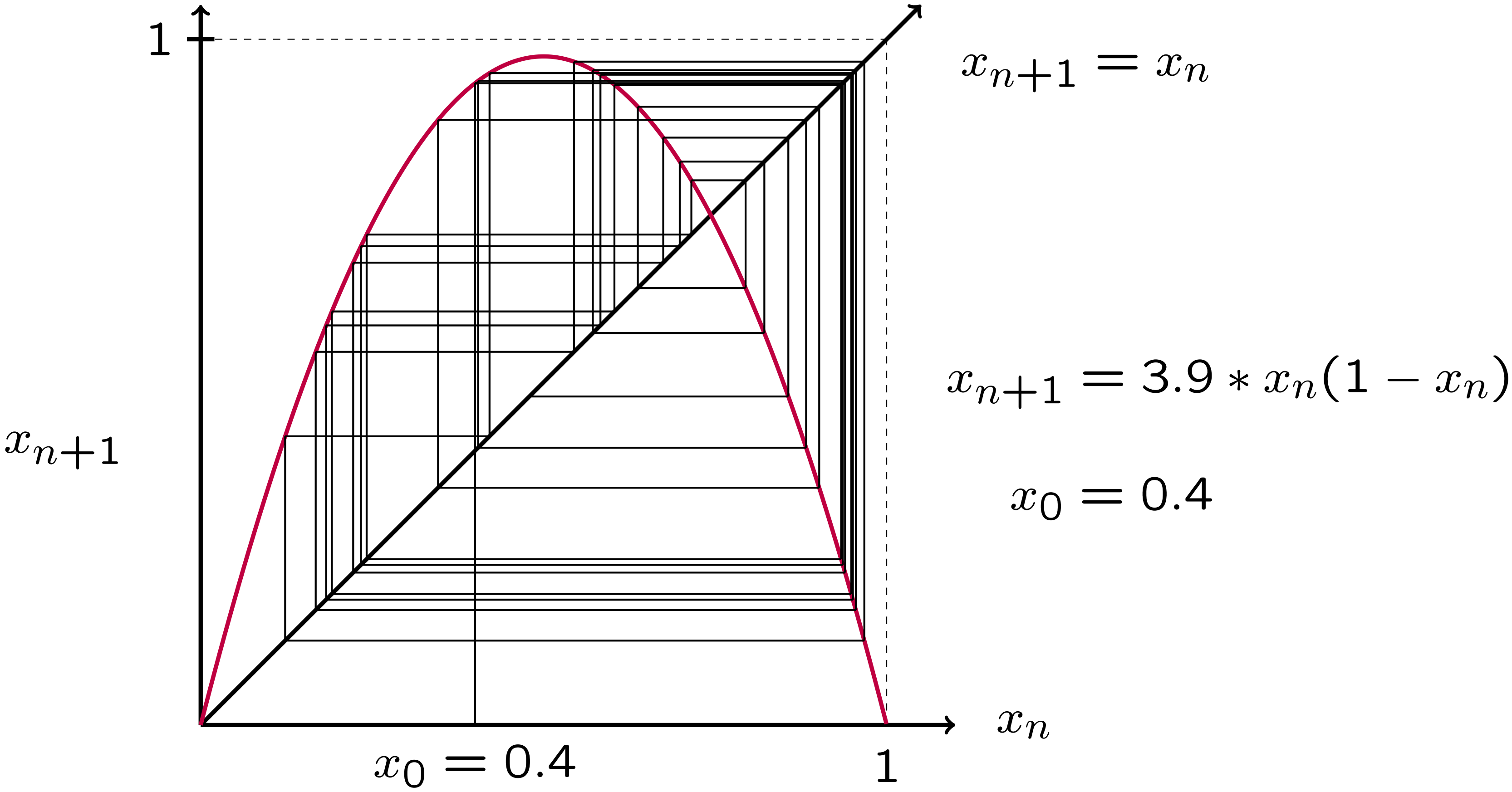
Let's view the behavior of the system in a somewhat different way. We're going shift over into a form of *phase space* for this system, where we will plot x_{n+1} against x_n. Note that for any value of $R > 0$, the right hand side is a parabola opening down, with roots at $x_n = 0$ and $x_n = 1$. The maximum value of the parabola occurs at $x_n = \frac{1}{2}$, and the maximum value is $\frac{R}{4}$.



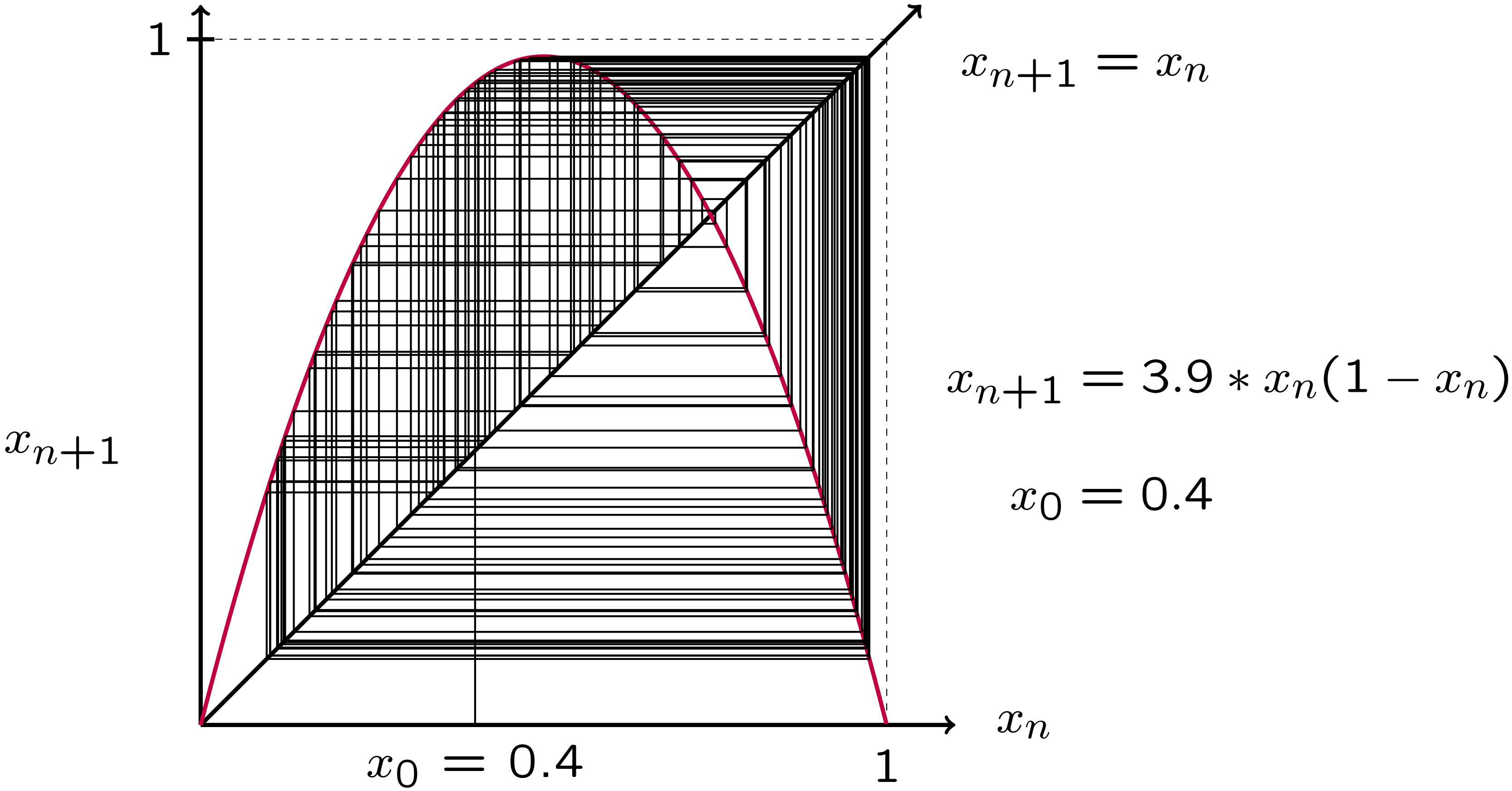
Now we'll follow the trajectory of the system. We'll visualize this by drawing a sequence of lines (often called a cobweb diagram). The coordinates of the endpoints of the lines will be $(x_0, 0)- > (x_0, x_1)$, $(x_0, x_1)- > (x_1, x_1)$, $(x_1, x_1)- > (x_1, x_2)$, $(x_1, x_2)- > (x_2, x_2)$, etc.



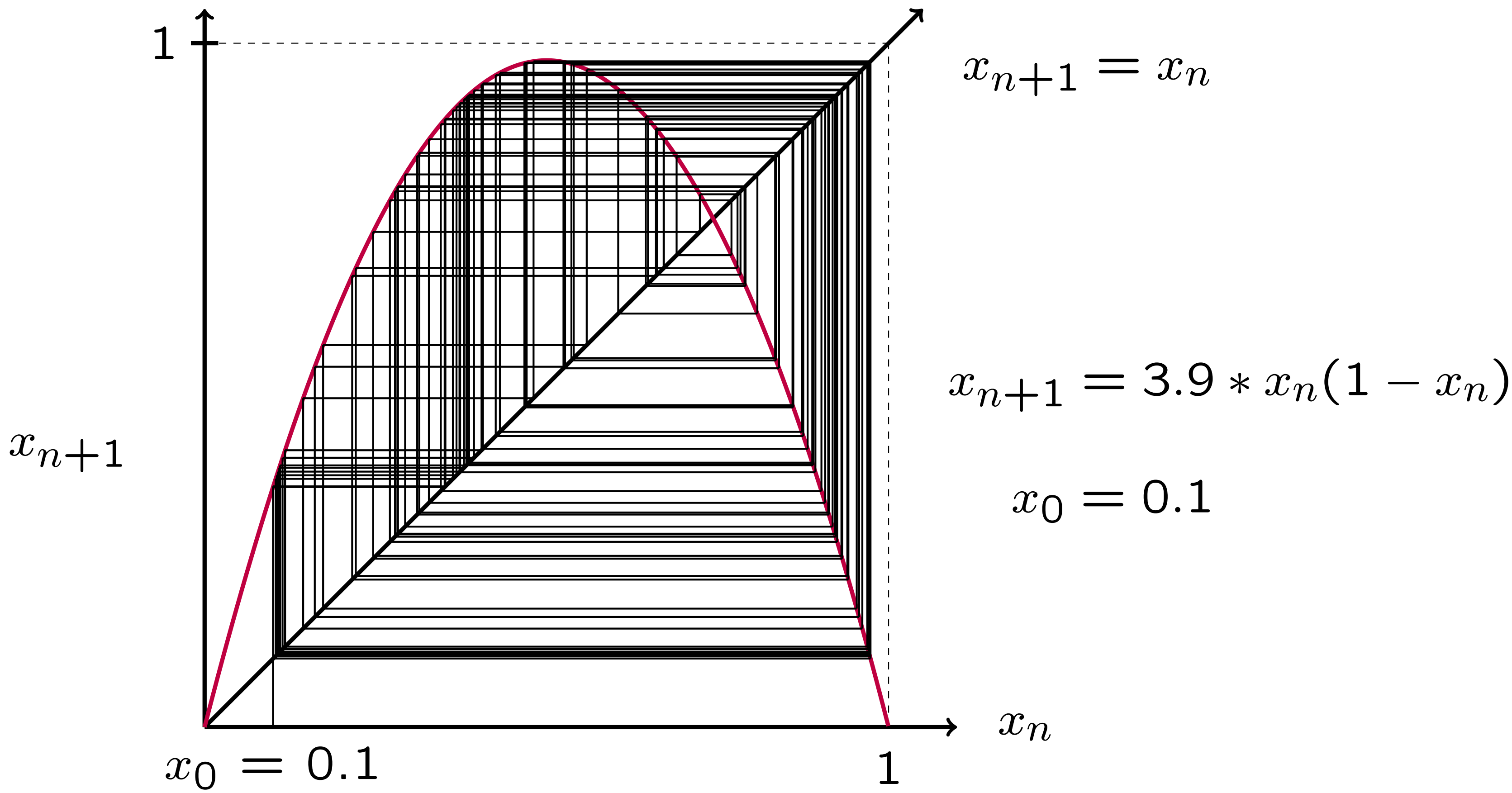
Let's let that same example run for a while:



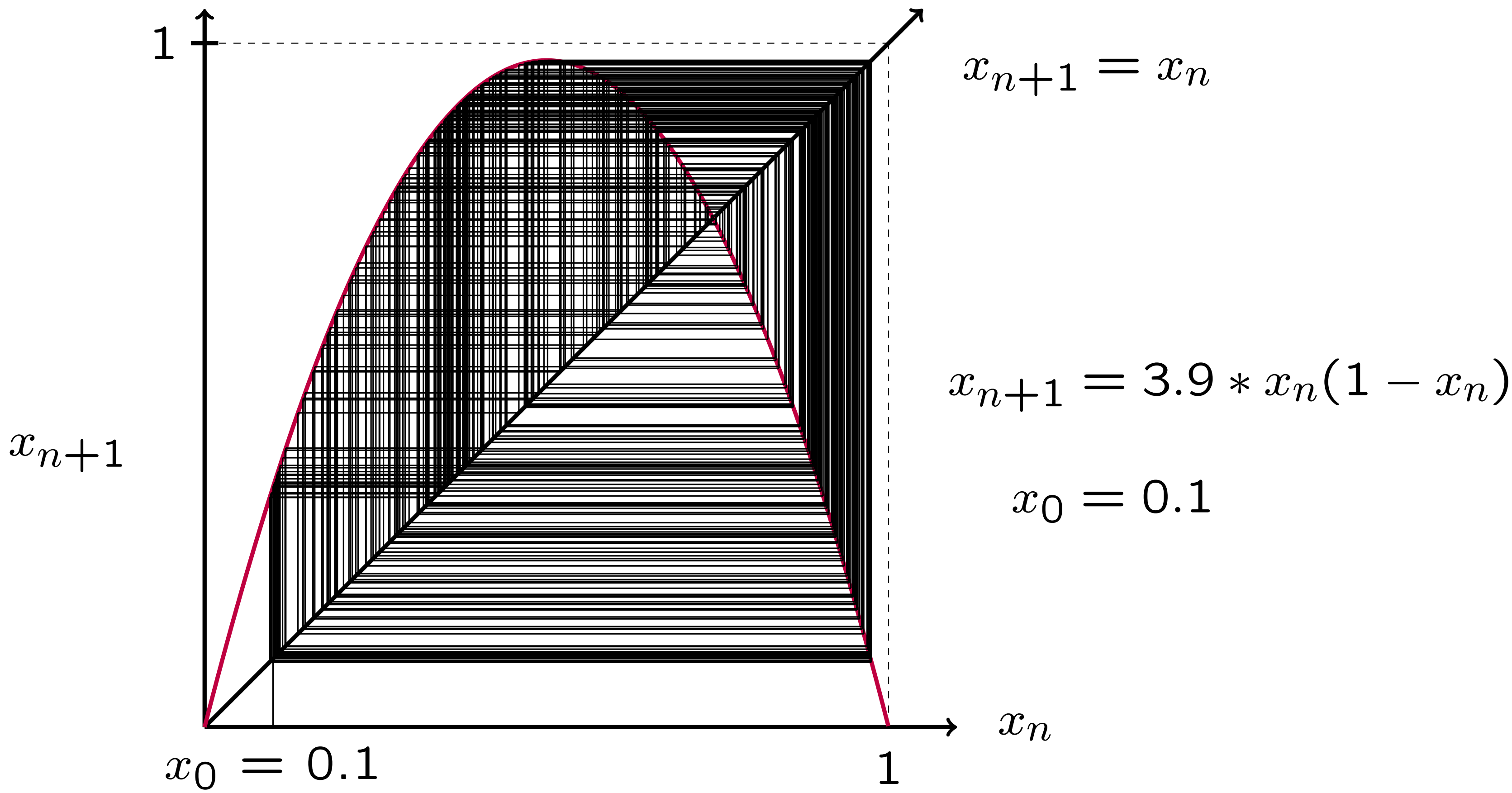
And even longer:



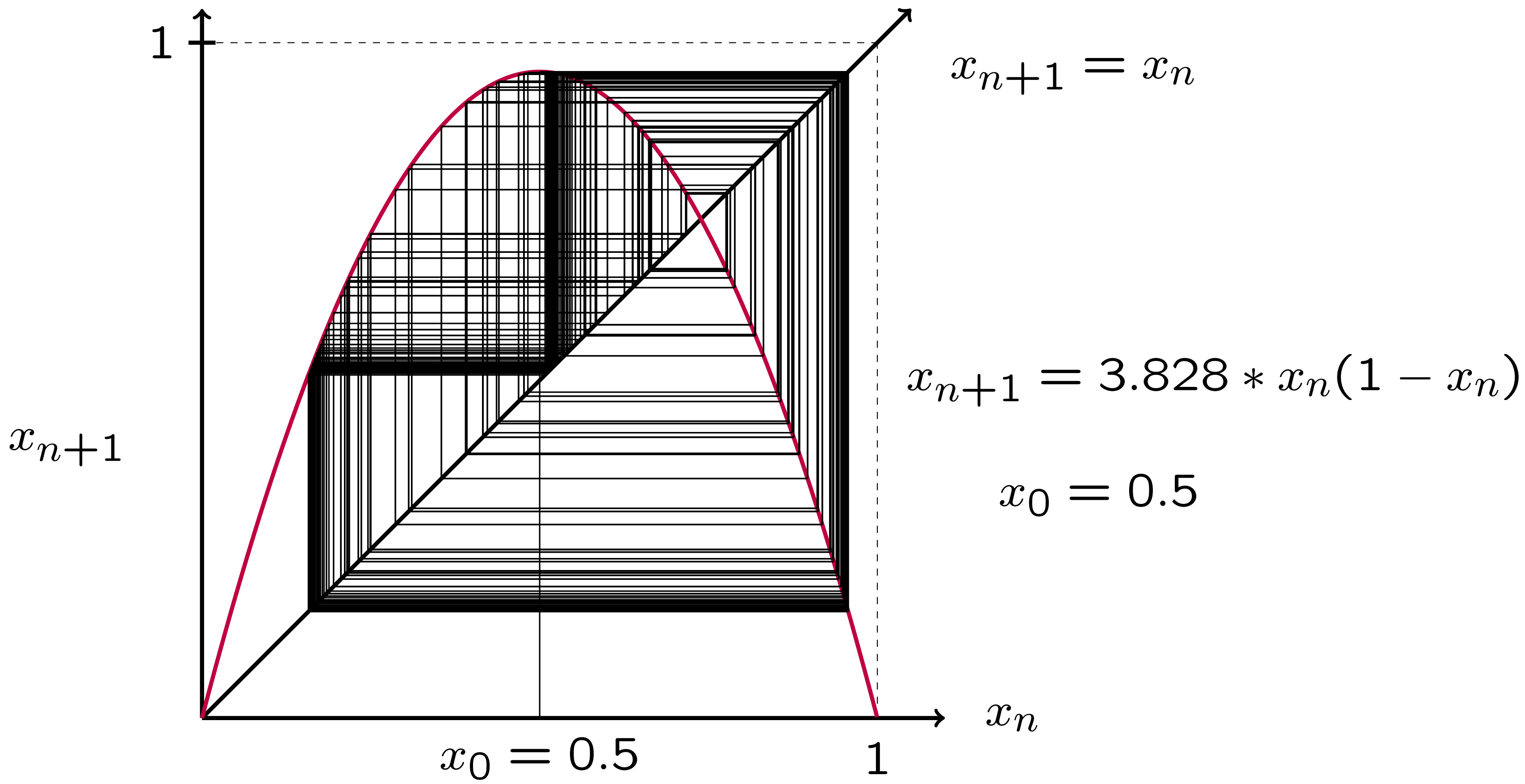
With a different starting value:



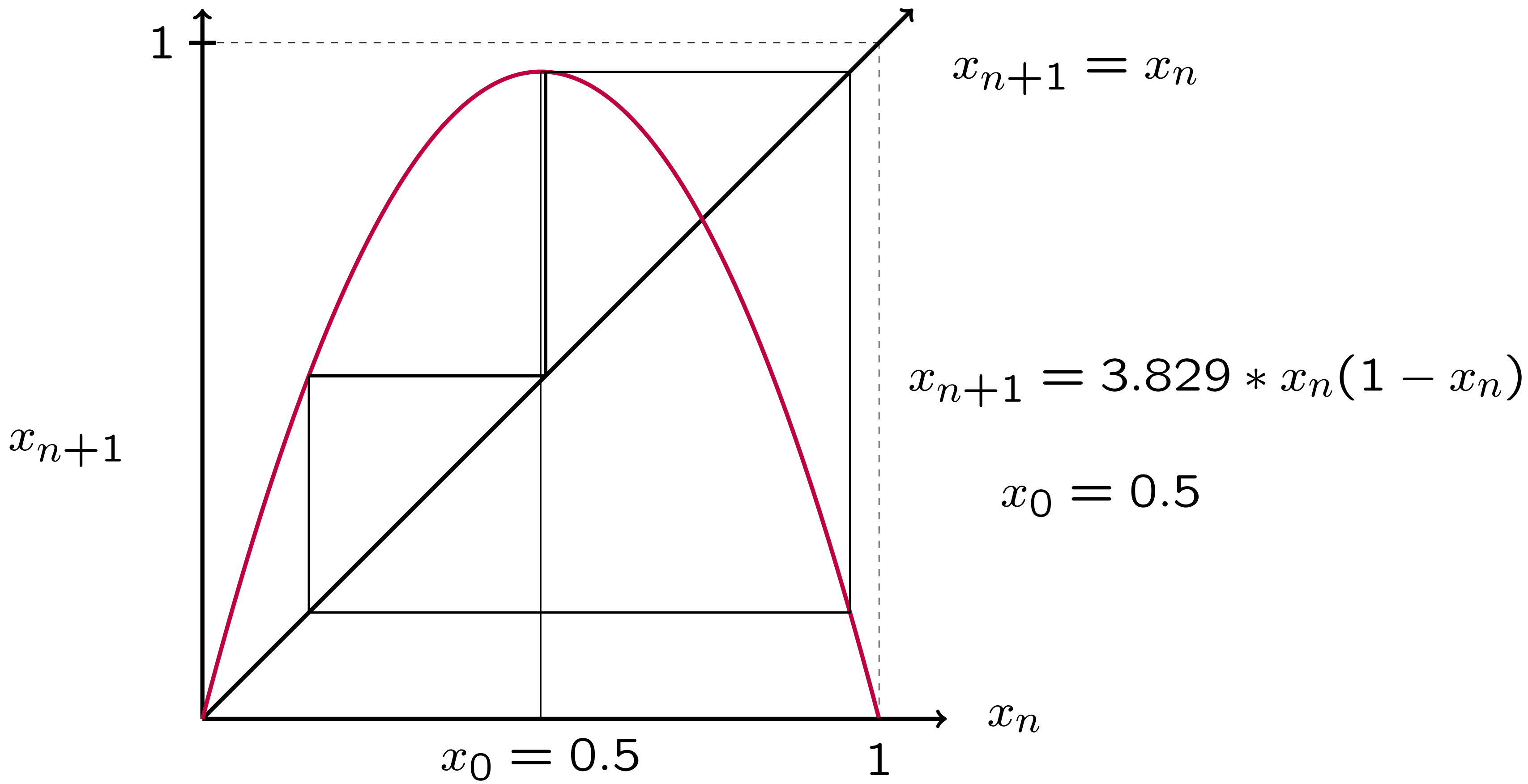
And even longer:



For other values of R (320 iterations):

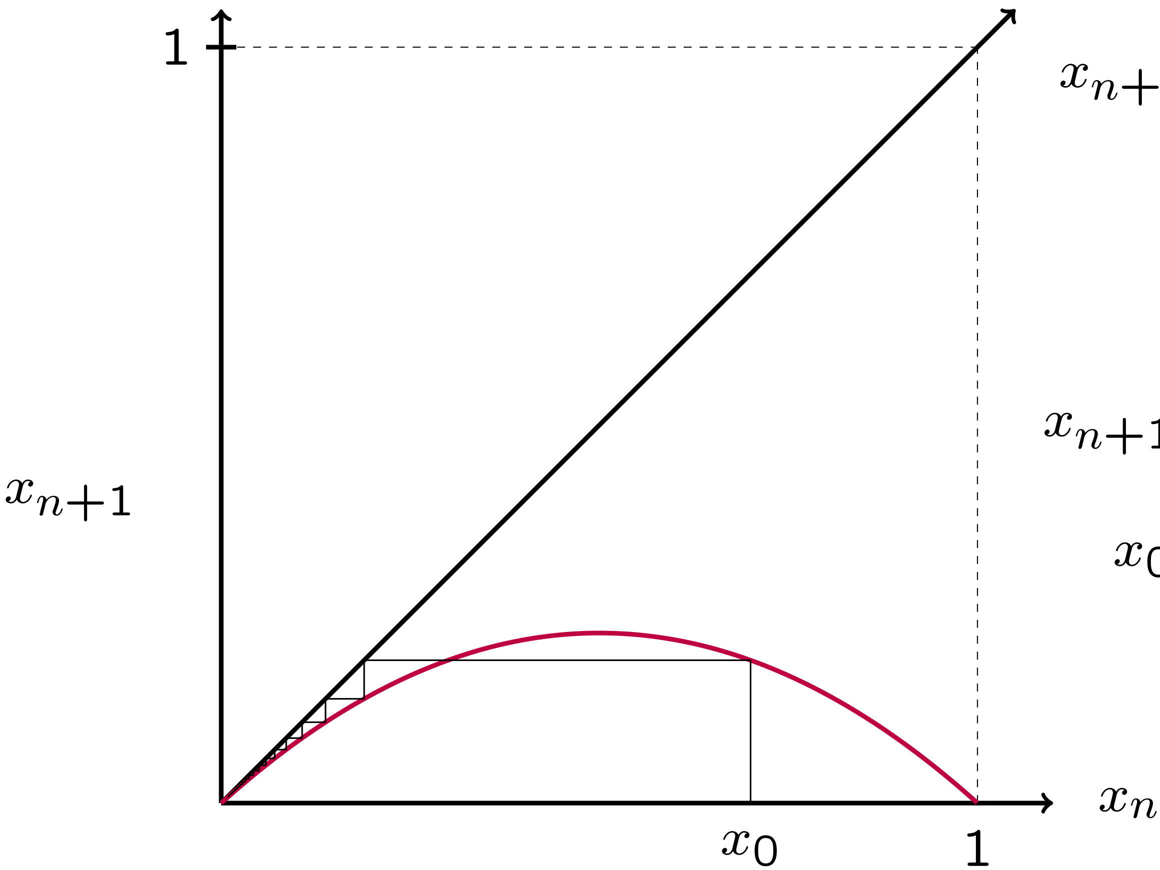


And slight changes (also 320 iterations):



- Now we'll be a little more systematic in our analysis. We'll start with small values of R, where the dynamics are relatively simple. Let's look at the slope of the tangent line to the parabola at 0. We are working with the parabola $P(x) = R * x * (1 - x)$, or $P(x) = -Rx^2 + Rx$, so the derivative is $P'(x) = -2Rx + R$. We thus have $P'(0) = R$, and hence when $R \leq 1$, the entire parabola (for $x > 0$) lies below the line $x_{n+1} = x_n$. In this case, the dynamics just die away to zero (i.e., $\lim_{n\to\infty}(x_n) = 0$). In particular, 0 is an *attracting fixed point* of the system. (Note that for any value of R, 0 is a fixed point of the system.)

  For $R \leq 1$, the dynamics don't depend in any significant way on the starting value – almost immediately, the system begins decaying to 0, and continues directly there.



$$x_{n+1} = x_n$$

$$x_{n+1} = 0.9 * x_n(1 - x_n)$$

$$x_0 = 0.7$$

For $R > 1$, we have a new feature – the parabola crosses the line $x_{n+1} = x_n$ at some point $x > 0$. In particular, we solve for that crossing point by setting $x_n = x_{n+1}$ – in other words, we have

$$x_n = R * x_n * (1 - x_n),$$

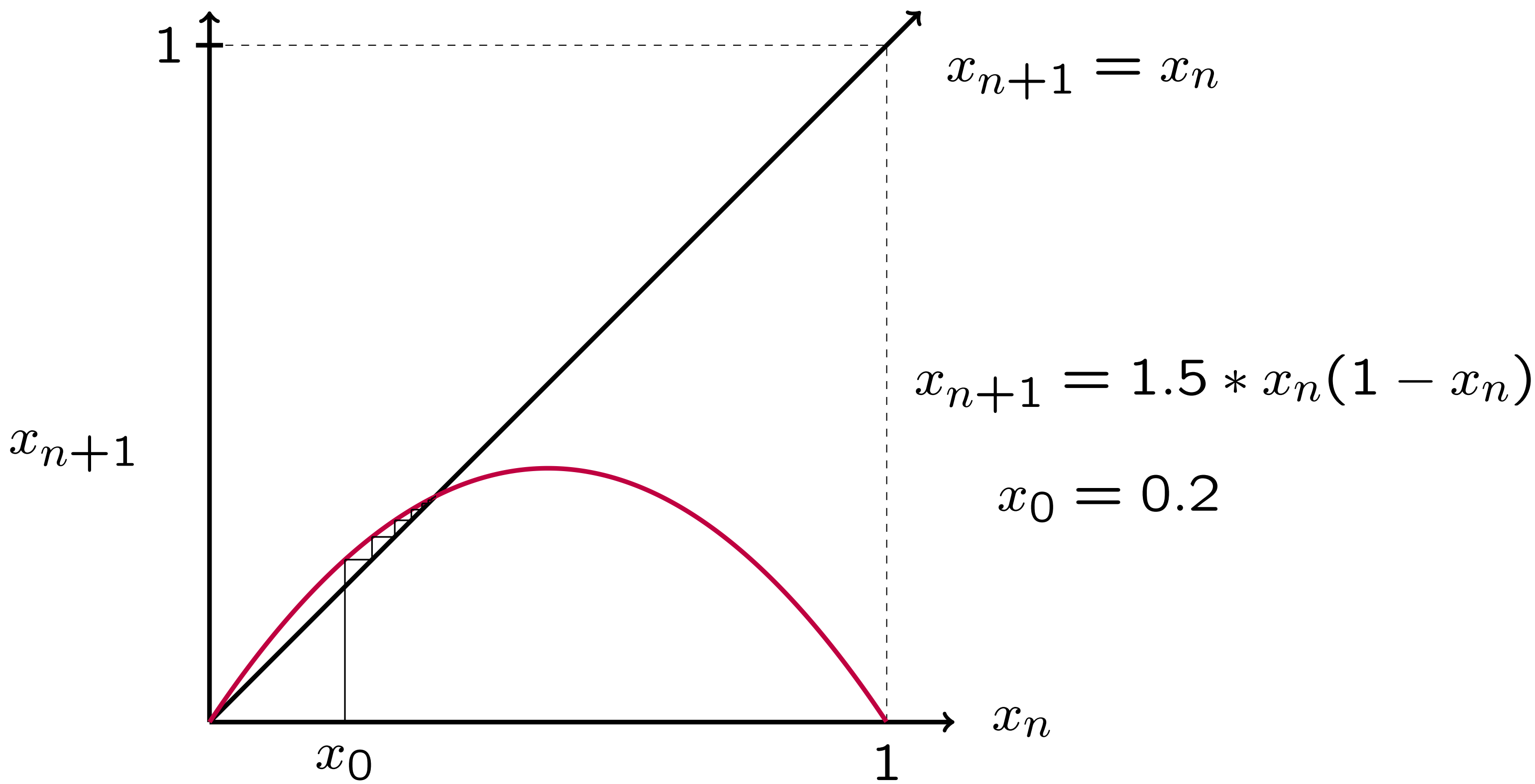
and so

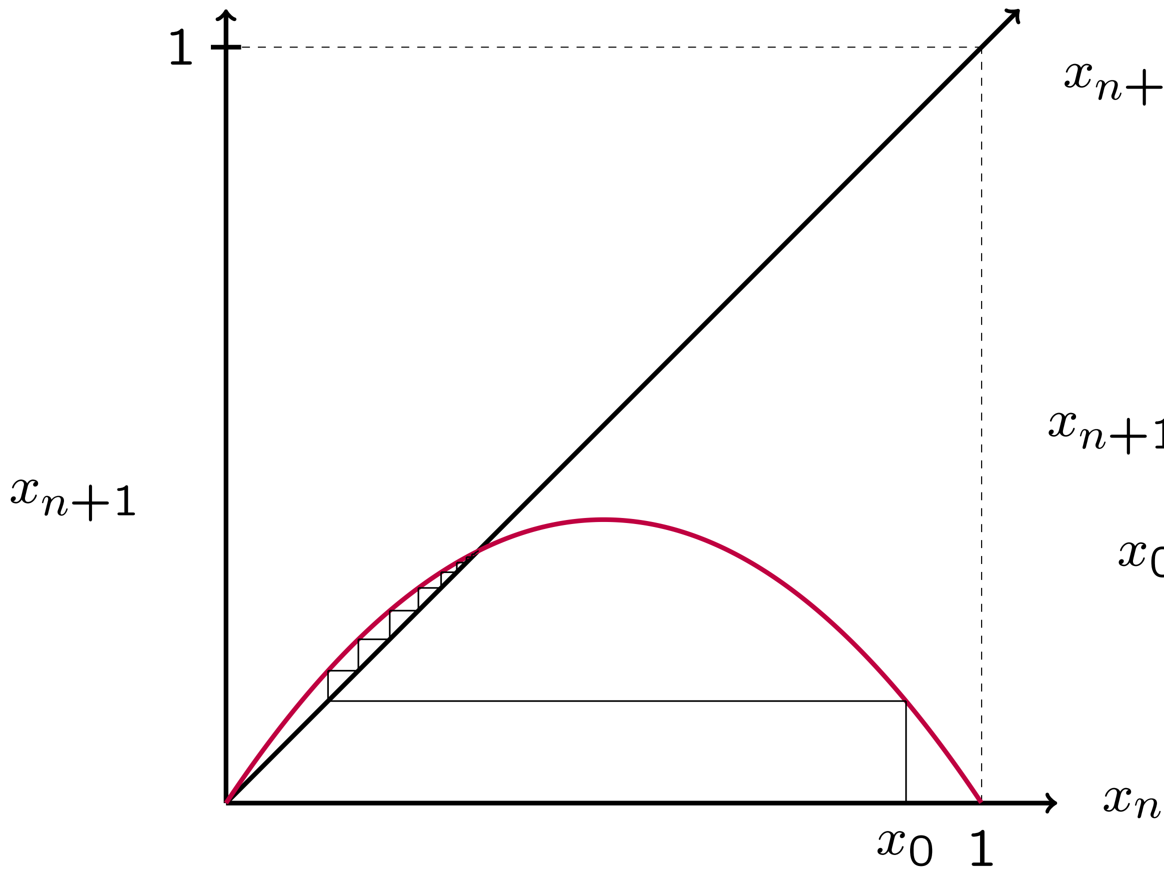
$$\begin{aligned} x_n - Rx_n(1 - x_n) &= 0 \\ x_n - Rx_n + Rx_n^2 &= 0 \\ Rx_n^2 + (1 - R)x_n &= 0 \\ x_n(Rx_n + 1 - R) &= 0, \end{aligned}$$

and hence the crossing point occurs at

$$x_n = \frac{R - 1}{R}.$$

For all values of $R > 1$, this crossing point is a fixed point of the system. For values $1 < R < 3$, this is an attracting fixed point for any starting value x_0 (except $x_0 = 0$ or 1, and 0 is now a *repelling* fixed point):

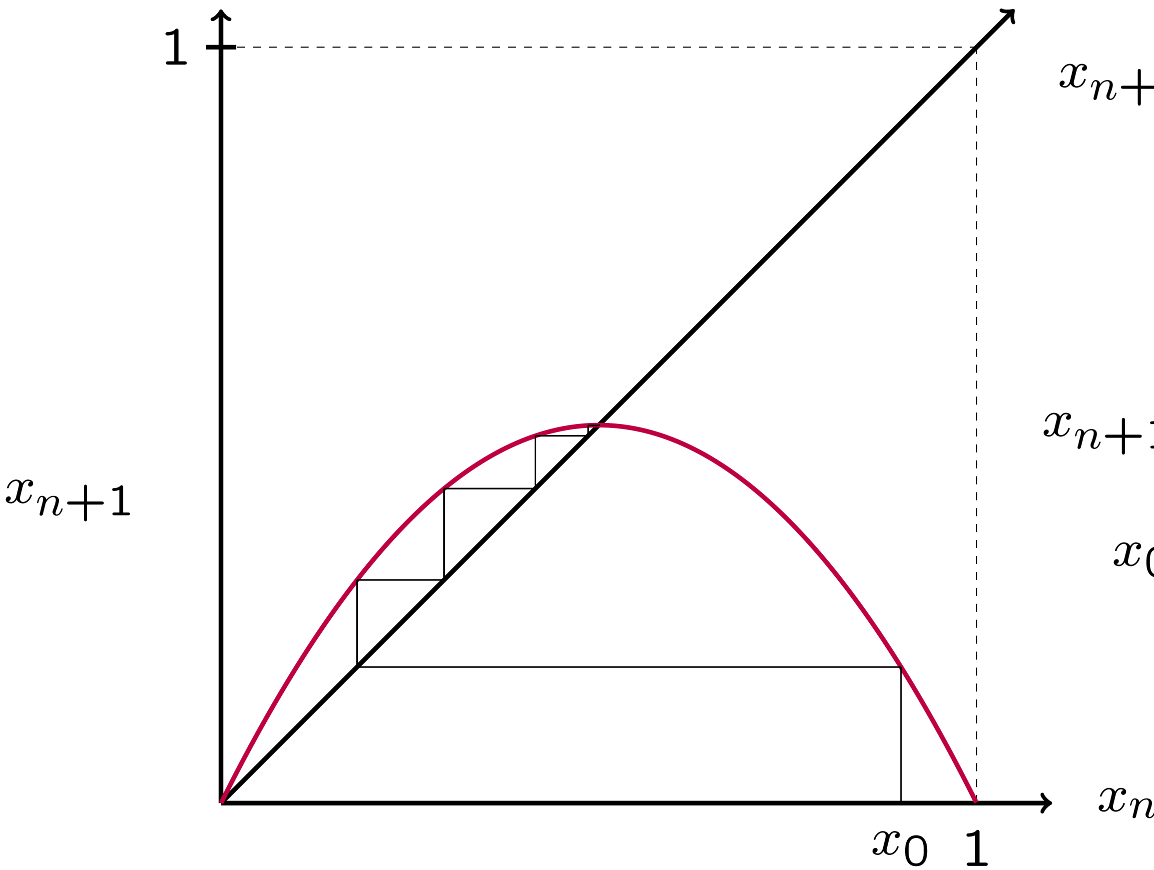




$x_{n+1} = x_n$

$x_{n+1} = 1.5 * x_n(1 - x_n)$

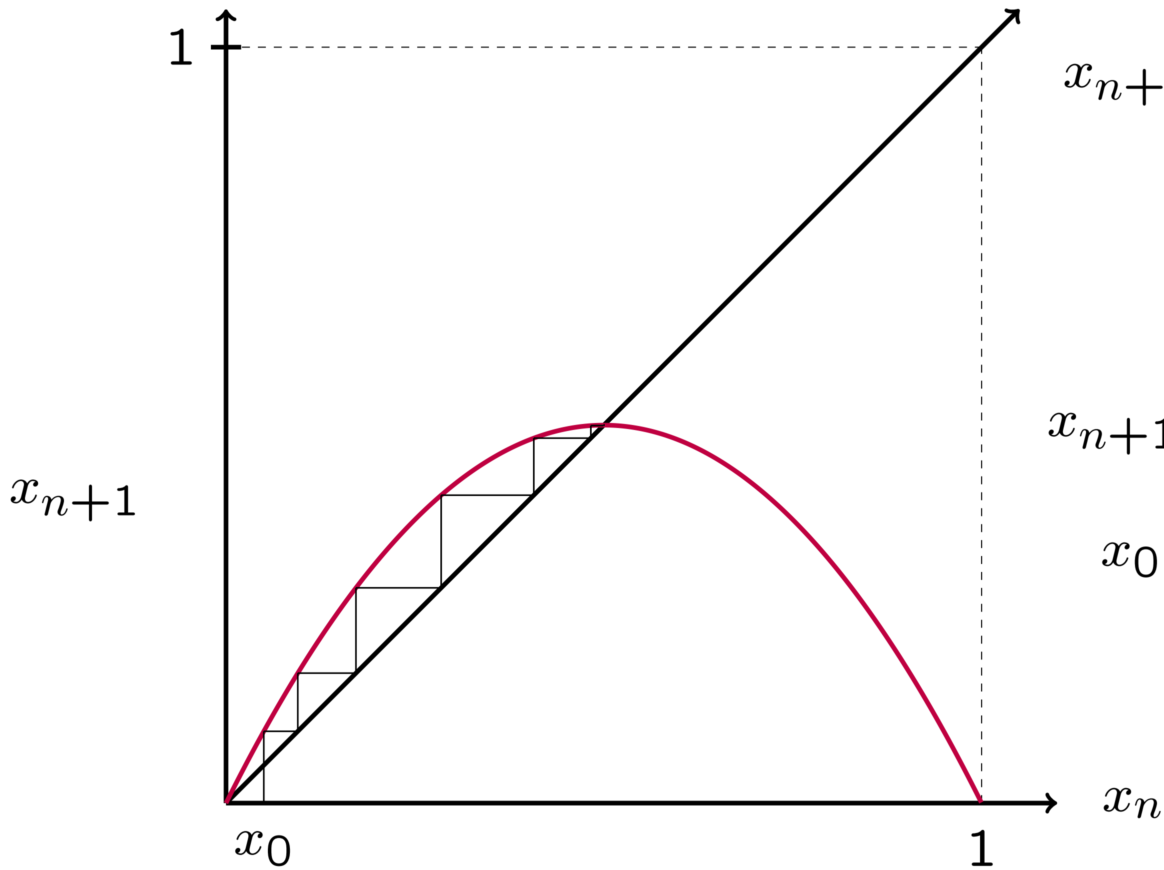
$x_0 = 0.9$



$$x_{n+1} = x_n$$

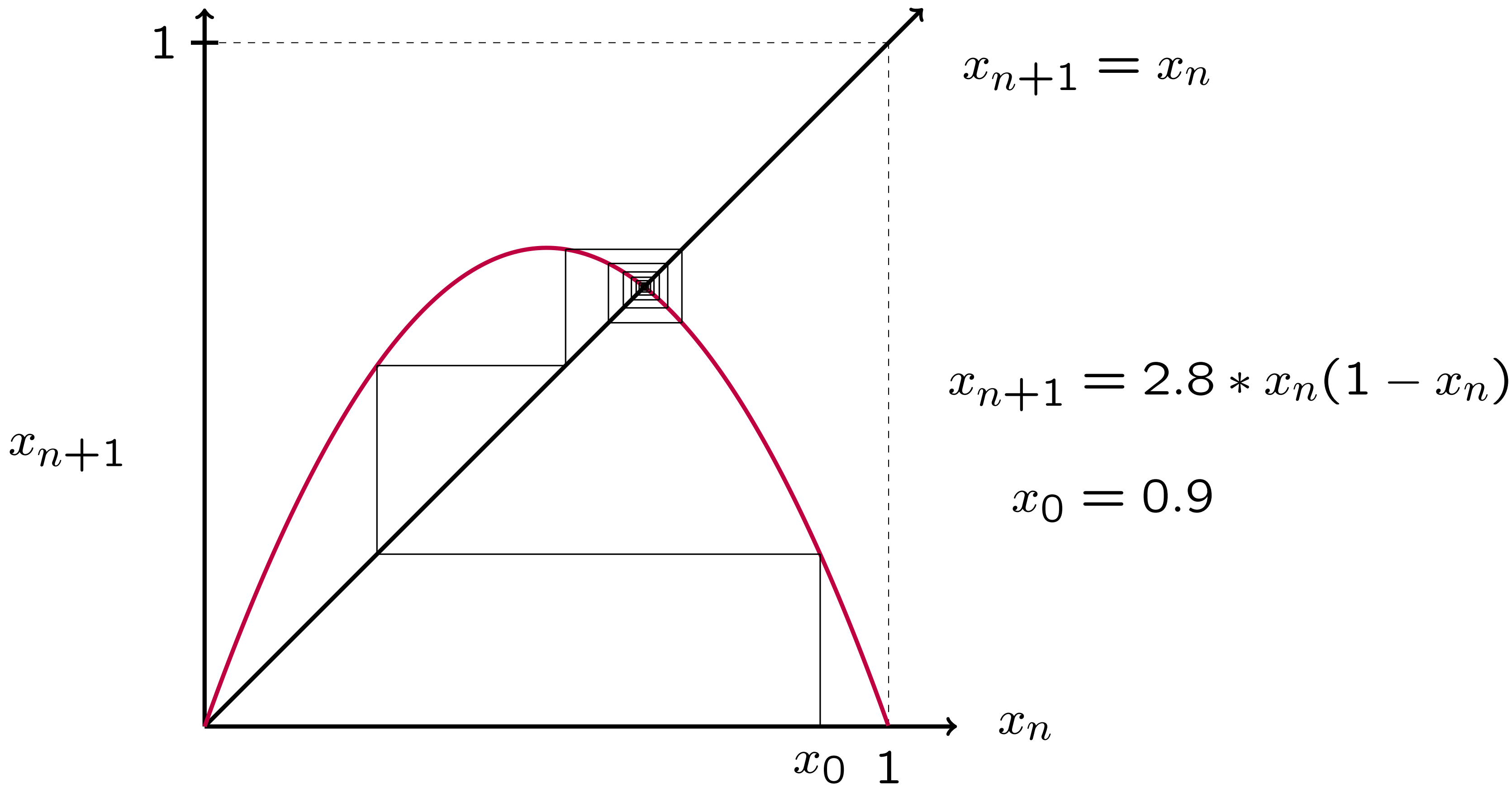
$$x_{n+1} = 2.0 * x_n(1 - x_n)$$

$$x_0 = 0.9$$



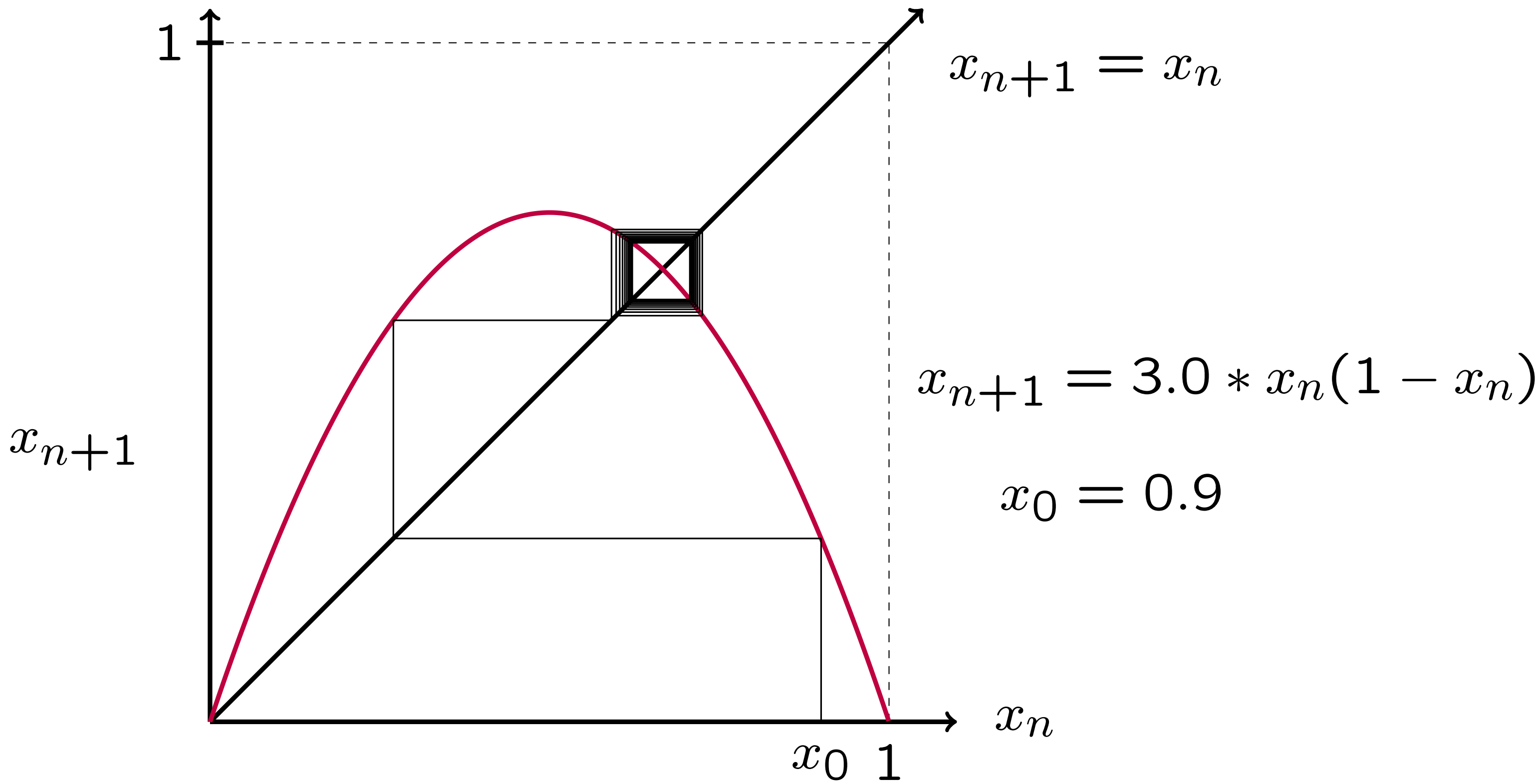
$$x_{n+1} = 2.0 * x_n(1 - x_n)$$

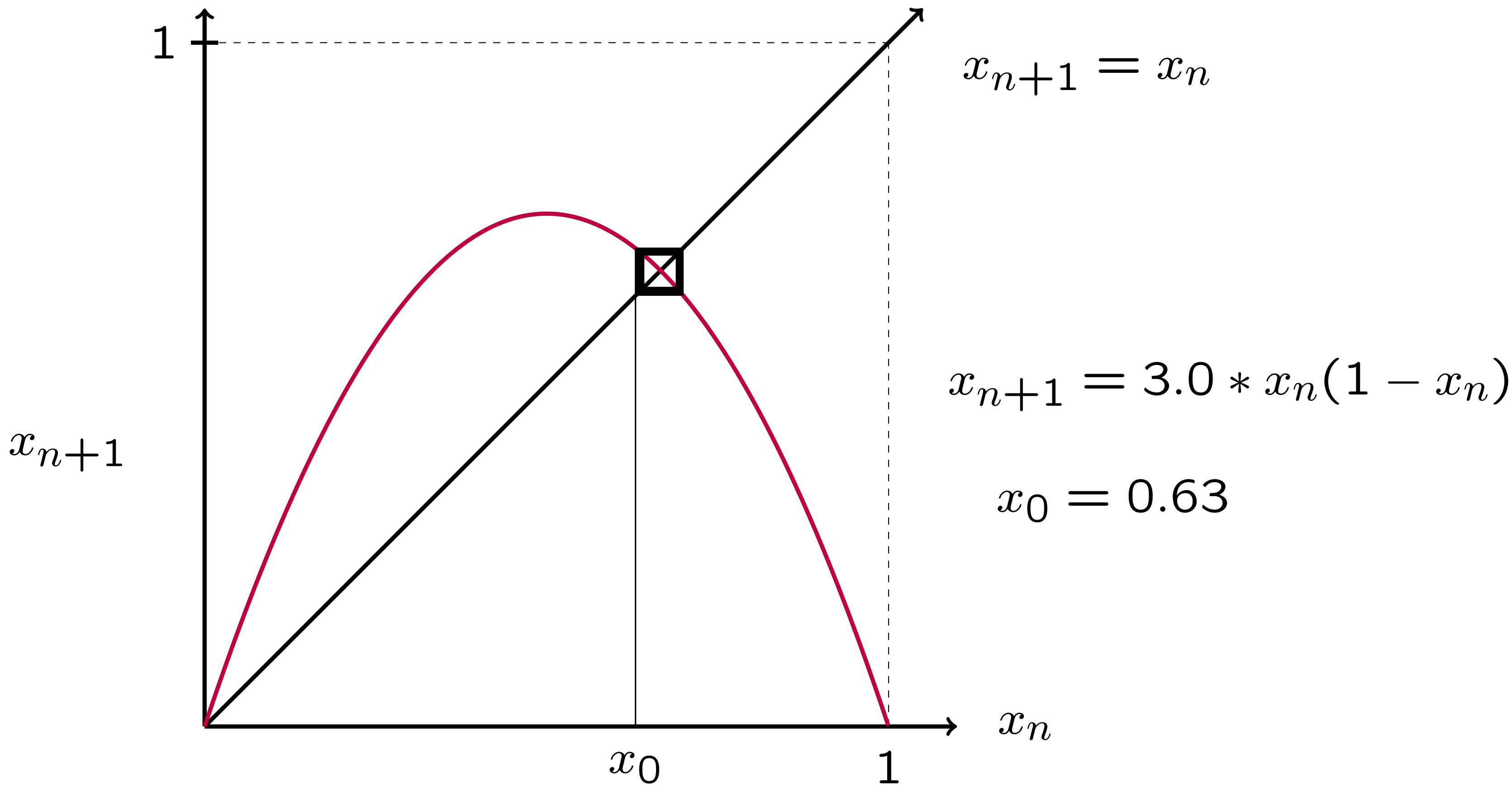
$$x_0 = 0.05$$



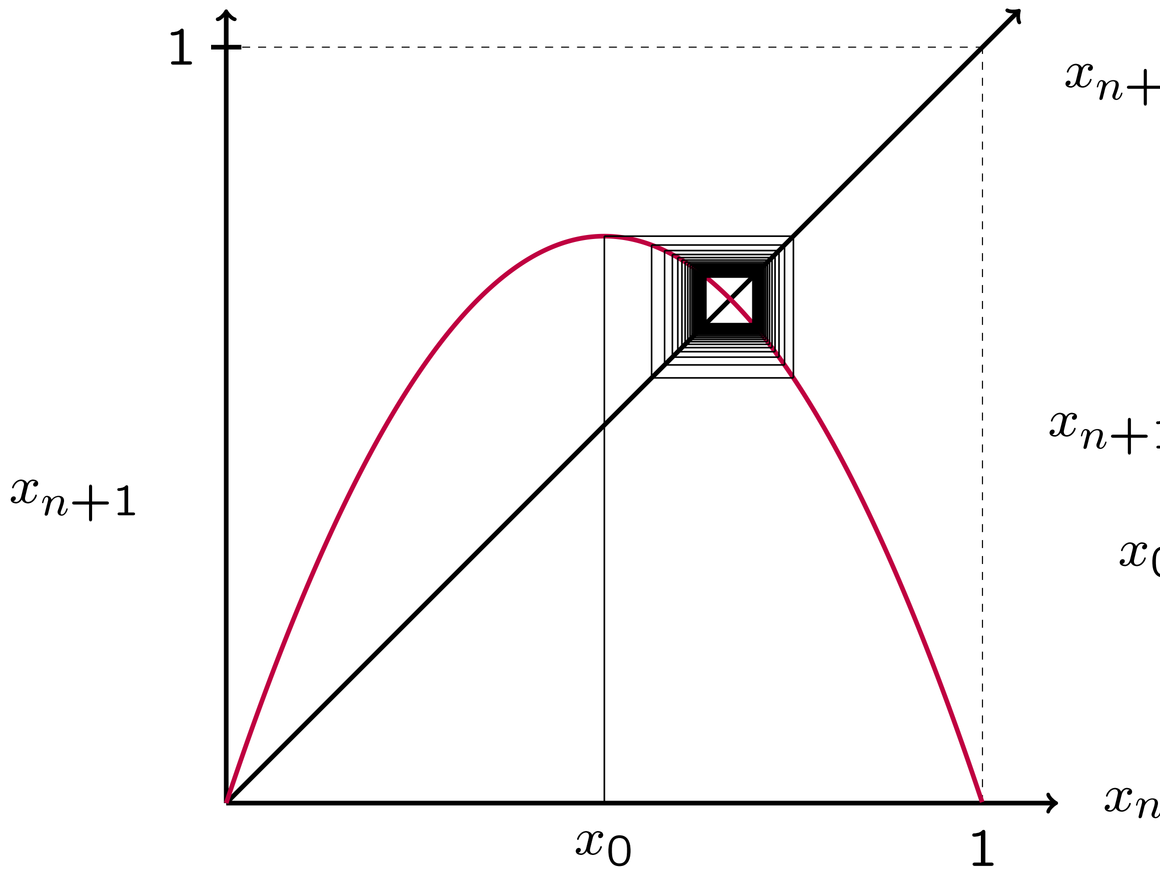
Note that for $2 < R < 3$, the parabola is coming back down at the fixed point $\frac{R-1}{R}$, and so the trajectory spirals in . . .

When $R = 3$, we have a new phenomenon. We still have a fixed point at $\frac{R-1}{R}$, but it is no longer and attracting fixed point:





1
$x_{n+1} = x_n$
$x_{n+1} = 3.0 * x_n(1 - x_n)$
$x_0 = 0.63$
x_{n+1}
x_n
x_0
1

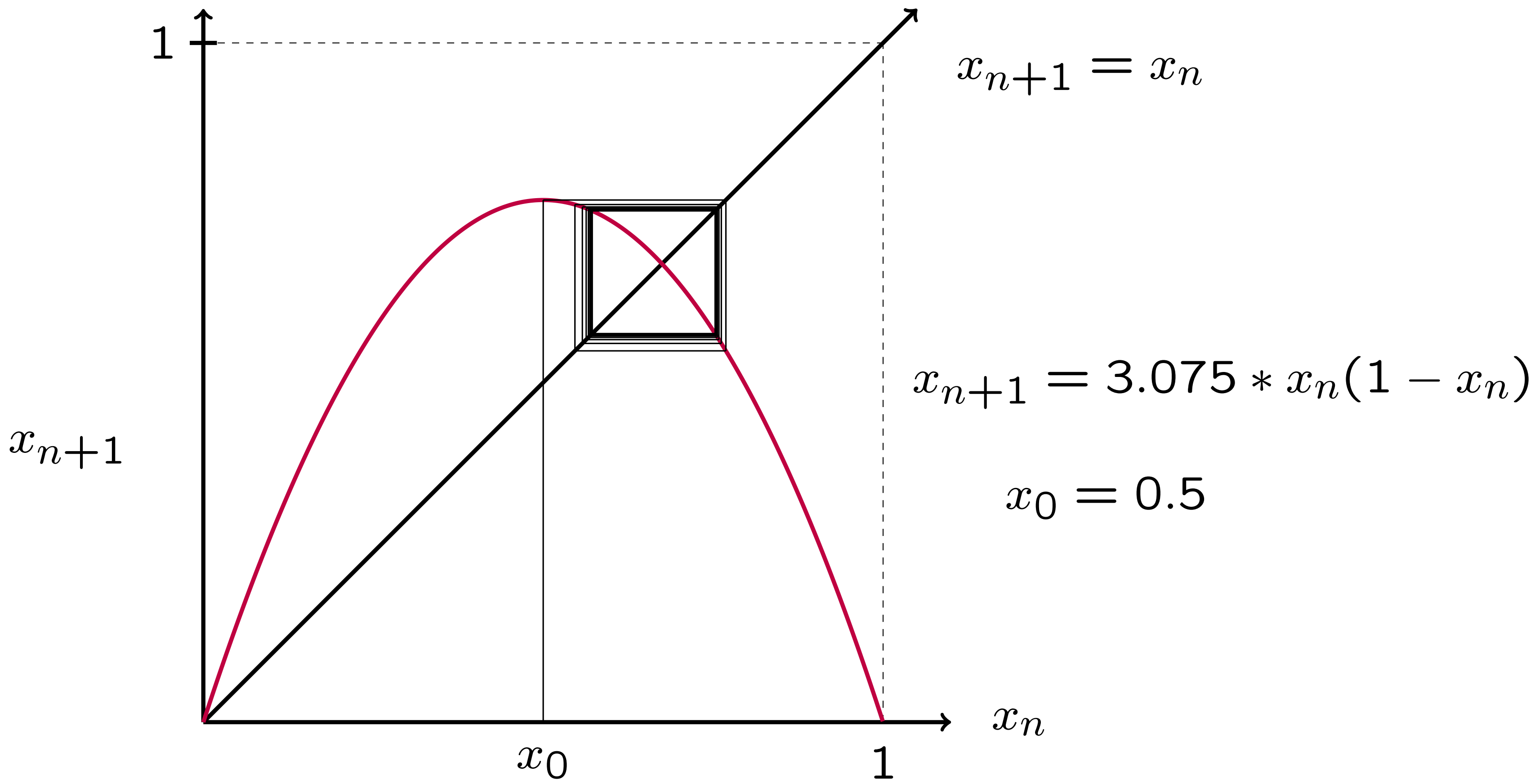


$$x_{n+1} = x_n$$

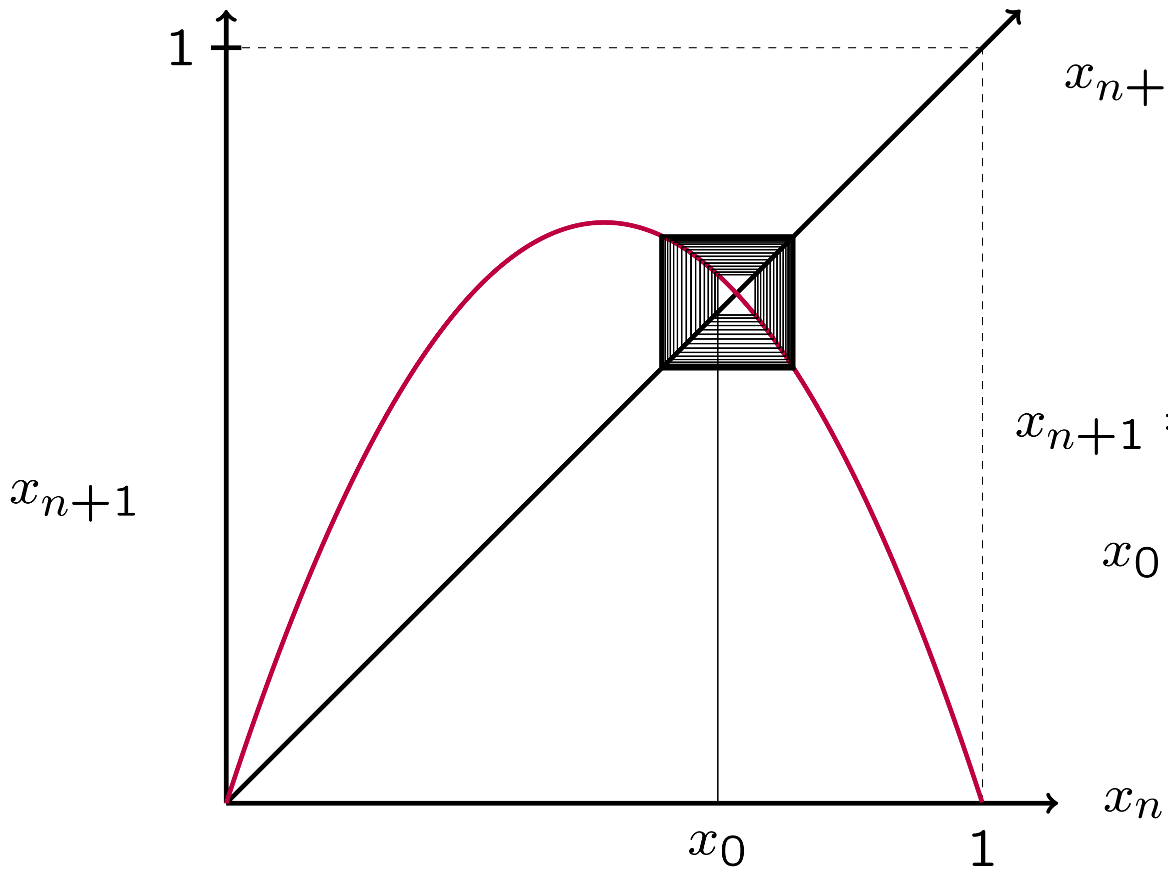
$$x_{n+1} = 3.0 * x_n(1 - x_n)$$

$$x_0 = 0.5$$

For R slightly bigger than 3, we have new behavior:



Or, starting closer to the fixed point:



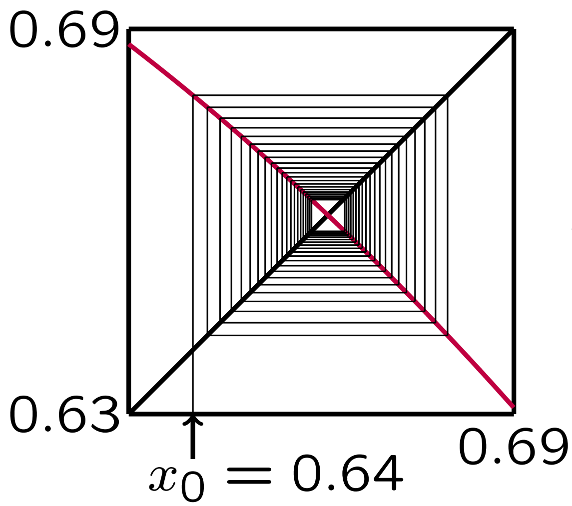
$x_{n+1} = x_n$

$x_{n+1} = 3.075 * x_n(1 - x_n)$

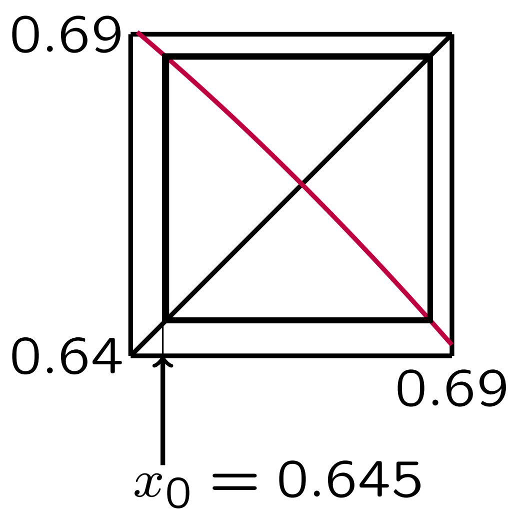
$x_0 = 0.65$

As R passes through 3, the fixed point goes from being an attracting fixed to point to being an *unstable fixed point* (note that we actually already observed this as R passed through 1, where 0 became an unstable fixed point).

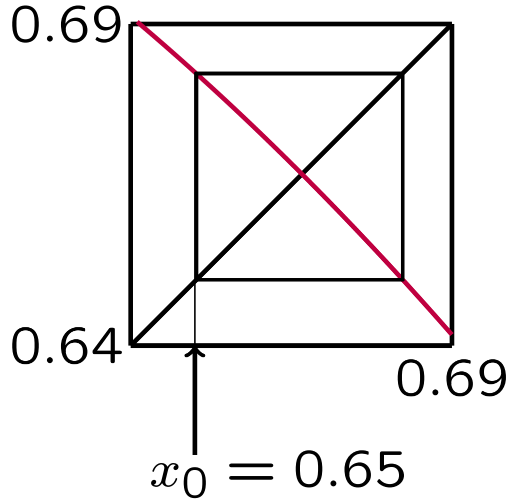
By looking more closely at the region of the fixed point as R passes through 3, we can understand better what is happening. In particular, the critical feature of the system is that as R goes through 3, the slope of the tangent to the parabola goes from above -1 to below -1 (i.e., becomes more steep).



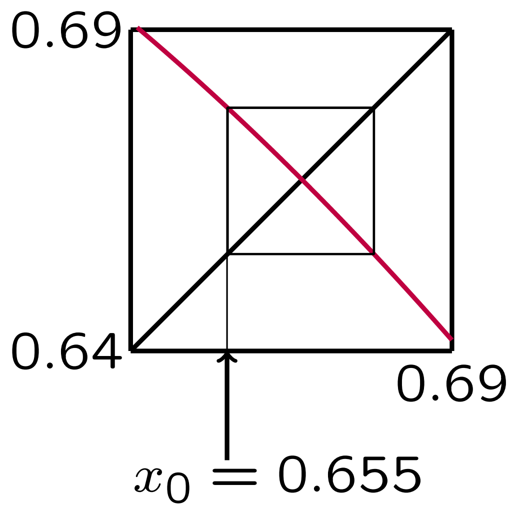
$$x_{n+1} = 2.95 * x_n(1 - x_n)$$



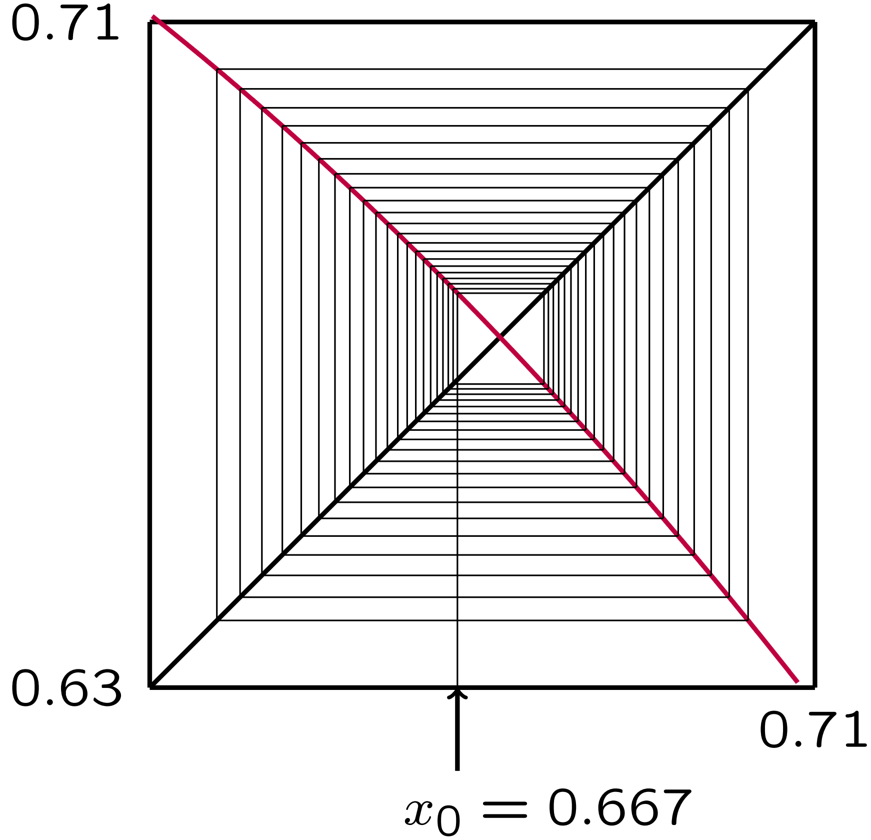
$$x_{n+1} = 3.0 * x_n(1 - x_n)$$



$$x_{n+1} = 3.0 * x_n(1 - x_n)$$

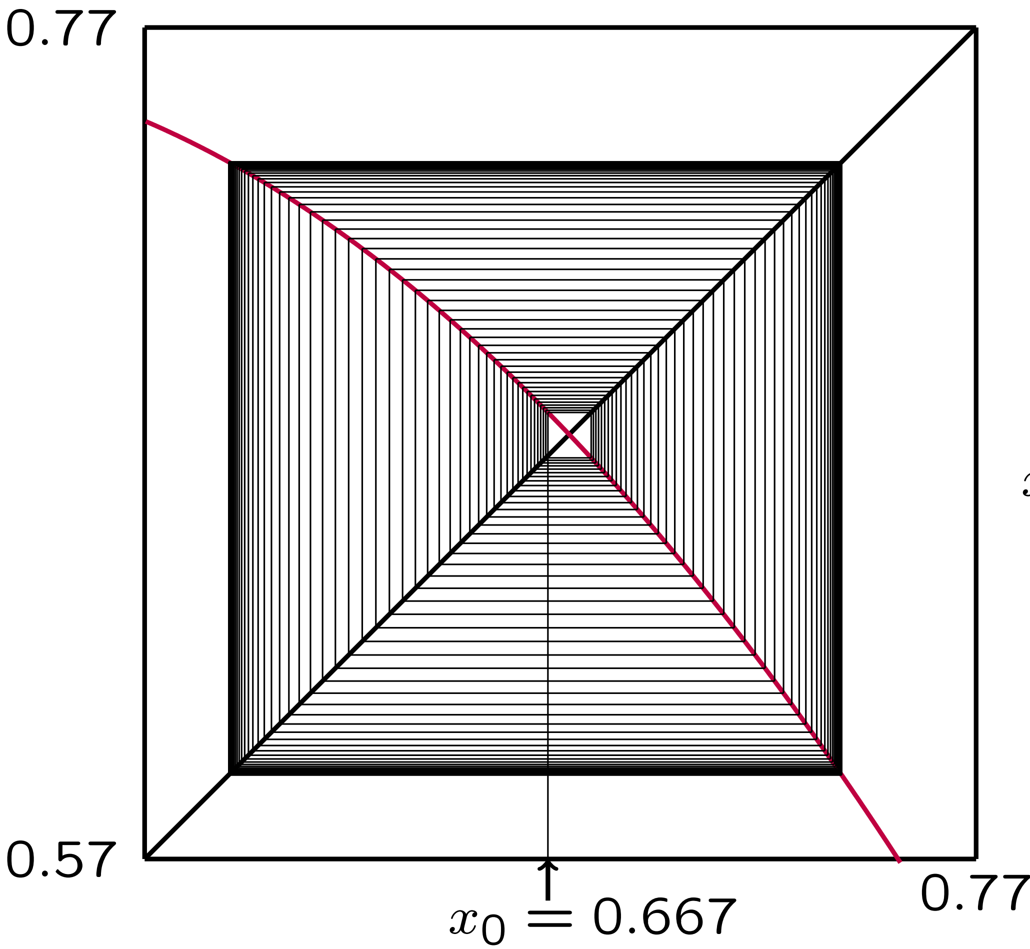


$$x_{n+1} = 3.0 * x_n(1 - x_n)$$

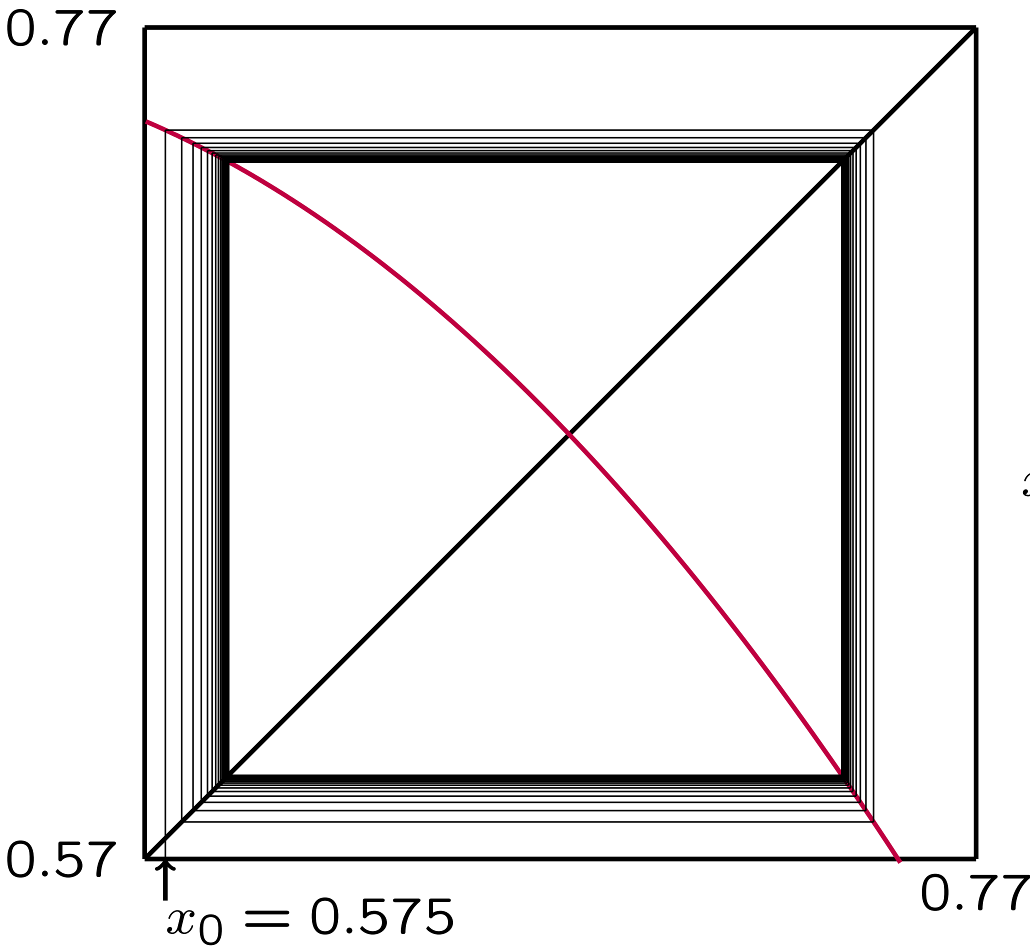


$$x_{n+1} = 3.05 * x_n(1 - x_n)$$

If we zoom out from this, we see another interesting phenomenon arise . . .



$$x_{n+1} = 3.05 * x_n(1 - x_n)$$



$$x_{n+1} = 3.05 * x_n(1 - x_n)$$

- We are seeing here an orbit of period 2 – the system bounces back and forth between two values. For values of $R < 3$ we had an attracting fixed point. The attracting fixed point is at 0 for $0 \leq R \leq 1$, and at $\frac{R-1}{R}$ for $1 < R < 3$. At $R = 3$ we still have the fixed point (and in fact that fixed point remains for all $R \geq 1$), but it is no longer a stable fixed point. For $R > 3$, it is a *repelling fixes point* – values near, but not exactly on, the fixed point will, in successive iterations, move further away.

  At $R = 1$ and $R = 3$, changes in R result in significant changes in the dynamics of the system. These significant changes in the dynamics of the system resulting from changes in a controlling parameter are called *bifurcations*. At $R = 1$, the stable (attracting) fixed point at 0 becomes unstable, and becomes a repelling fixed point. We acquire a new stable (attracting) fixed point at $\frac{R-1}{R}$.

At $R = 3$, the fixed point at $\frac{R-1}{R}$ becomes unstable, and we acquire a new attracting stable orbit of period 2. These is the first of a sequence of bifurcations in the dynamics of the logistics equation. At each of these bifurcations, an orbit of period 2^n becomes unstable (although it continues to exist in the dynamics), and a new stable orbit of period 2^{n+1} arises. This process is called a *period doubling bifurcation cascade*.

We can calculate the x values of this period 2 orbit by looking at two steps of the difference equation:

$$\begin{aligned}
x_{n+2} &= Rx_{n+1}(1 - x_{n+1}) \\
&= R(Rx_n(1 - x_n))(1 - Rx_n(1 - x_n)) \\
&= R^2 x_n(1 - x_n)(1 - Rx_n + Rx_n^2) \\
&= R^2 x_n(-Rx_n^3 + 2Rx_n^2 - (R + 1)x_n + 1)
\end{aligned}$$

We are looking for period 2 orbits – in other words, values where $x_{n+2} = x_n$. We therefore want

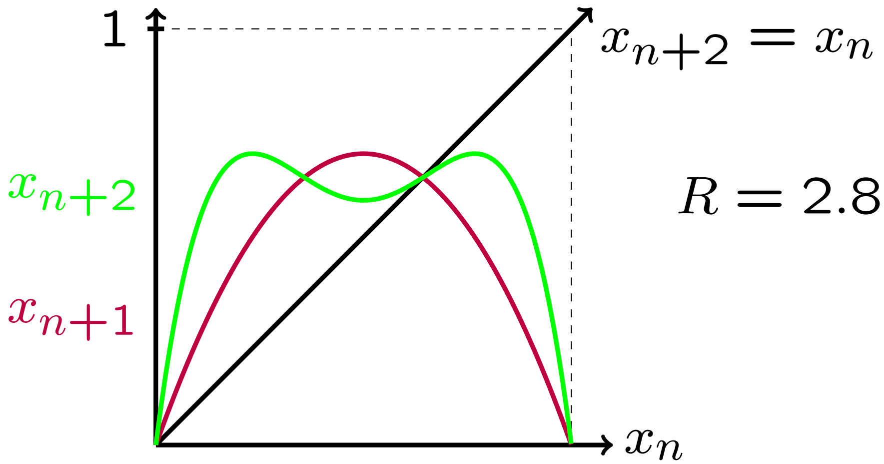
$$x_n = R^2 x_n(-Rx_n^3 + 2Rx_n^2 - (R+1)x_n + 1)$$

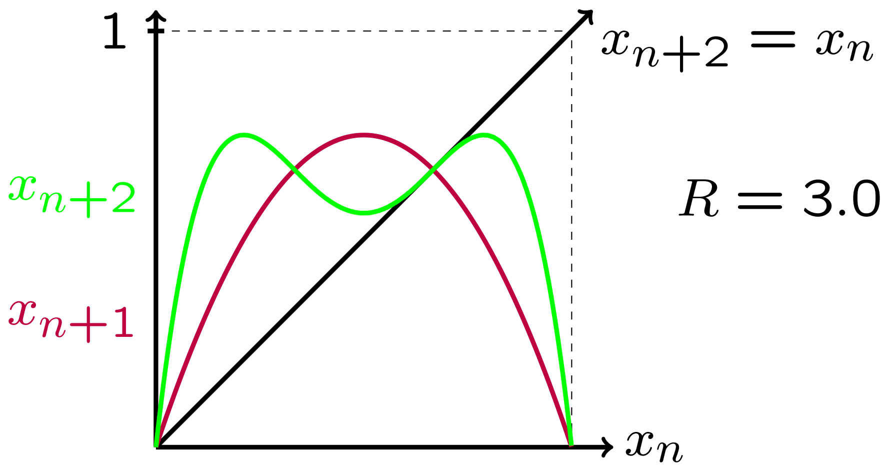
or

$$R^2 x_n(-Rx_n^3 + 2Rx_n^2 - (R+1)x_n + 1) - x_n = 0.$$

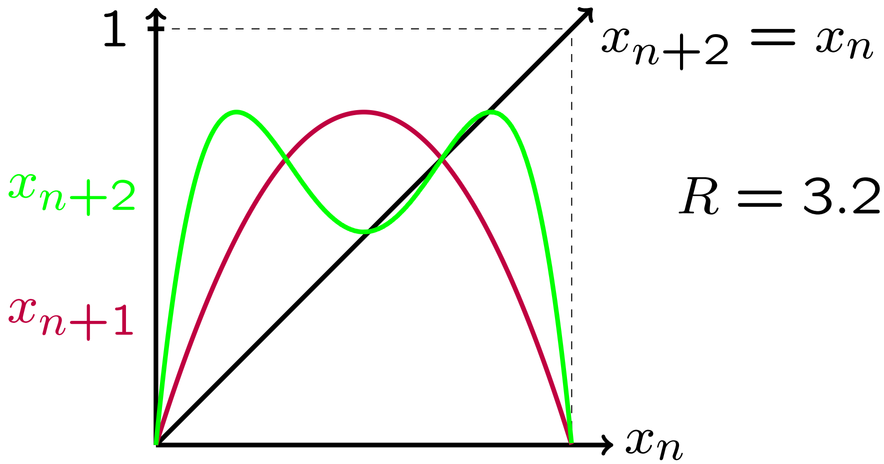
This is a 4th degree polynomial is x_n, which, for $R > 3$, has 4 roots. It is easy to see that $x_n = 0$ is a root (but we already knew that, because 0 is a fixed point of the original equation, and hence also repeats itself every 2 steps). Similarly, $x_n = \frac{R-1}{R}$ is a root, because it too is a fixed point. We could proceed with the (somewhat messy) algebra to find the other two roots, but instead let's look at the cobweb diagram for x_{n+2} vs. x_n (in these diagrams, we're also seeing the original parabolas, for reference . . .).

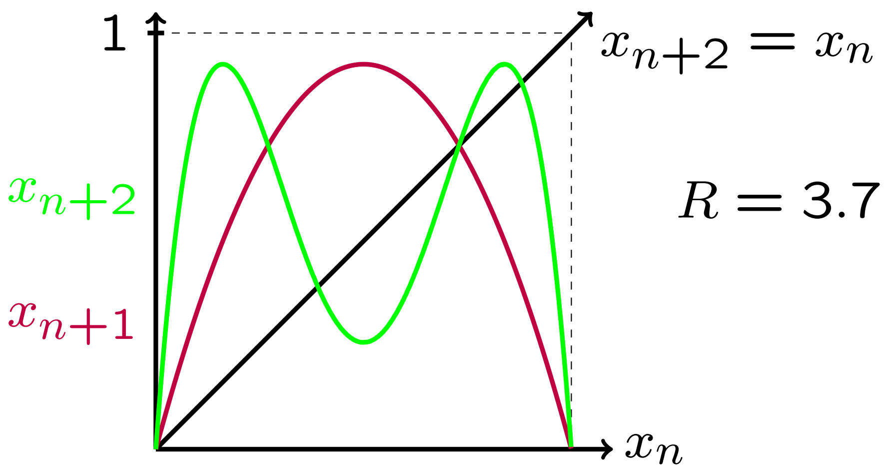
You can see the bifurcation happen as R reaches 3.0:



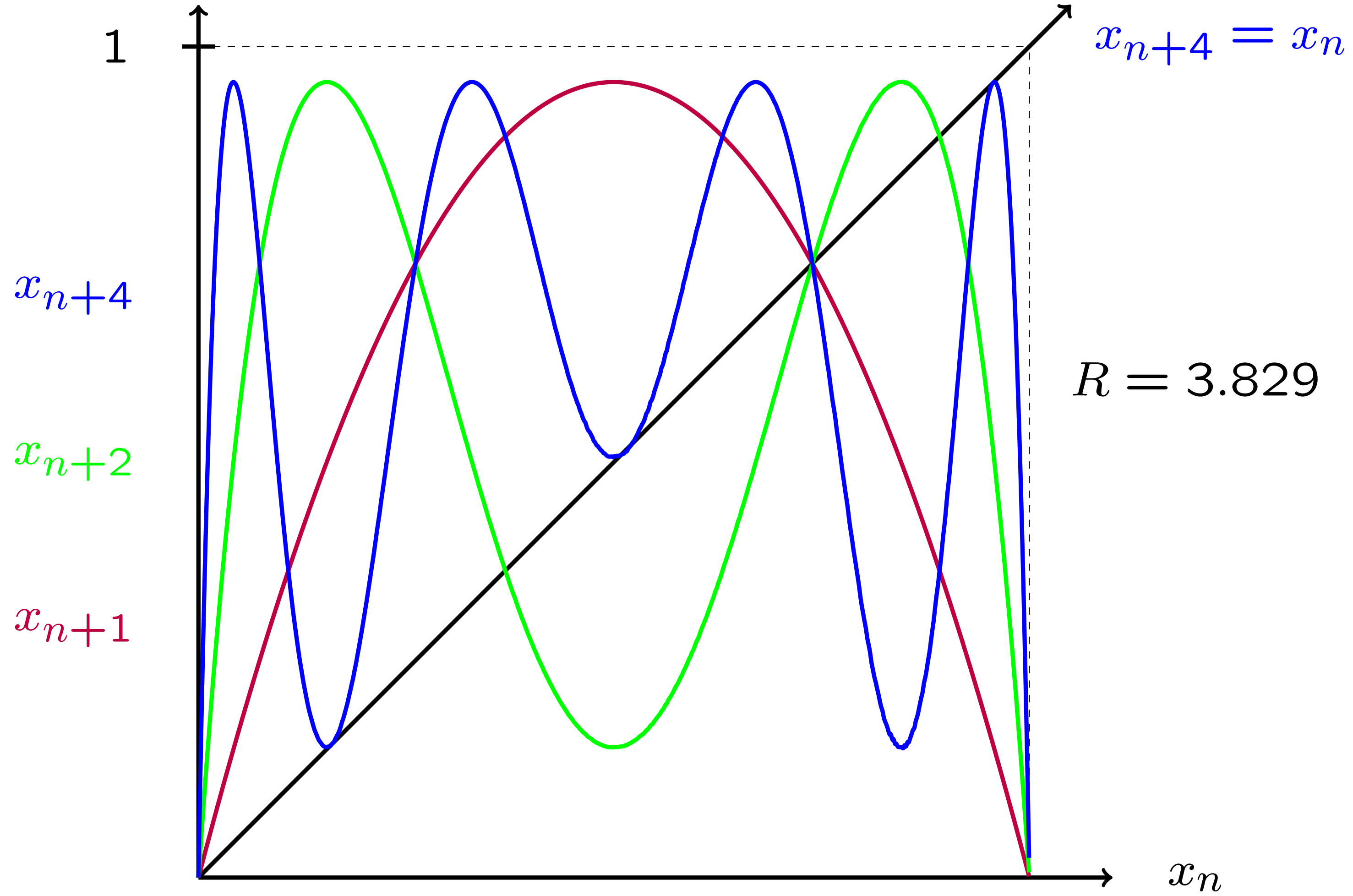


and moves on past . . .

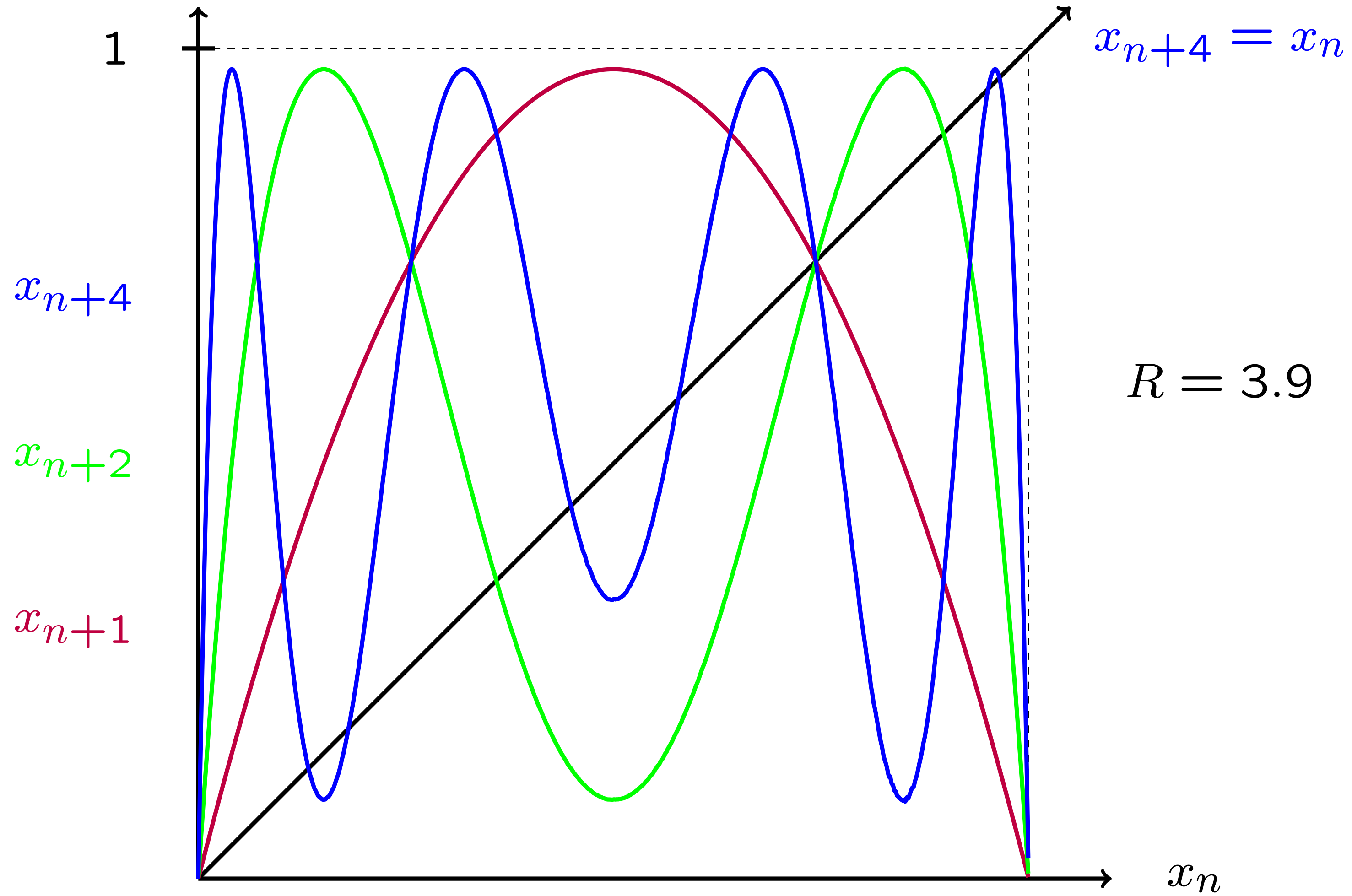




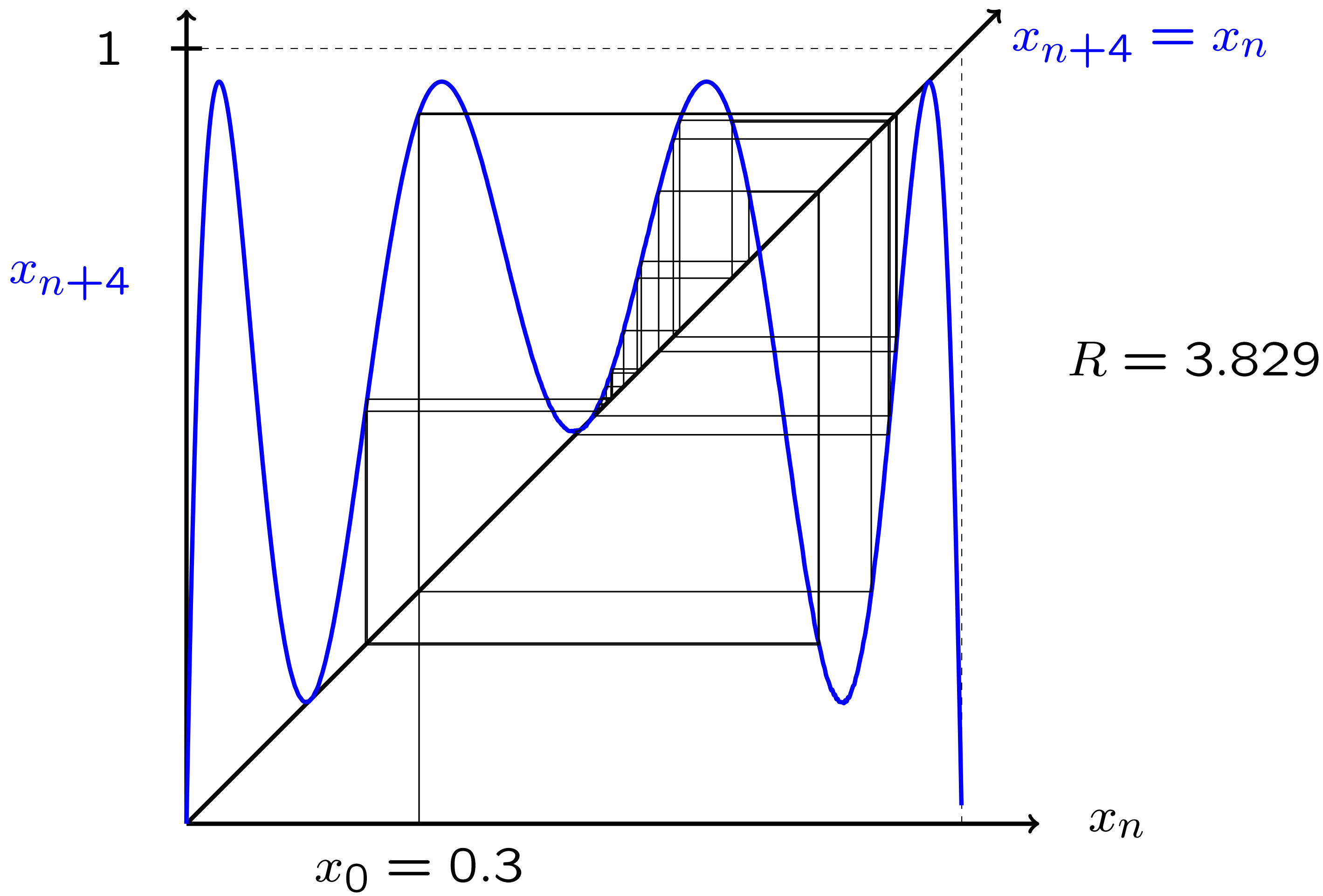
We can also look at higher order iterates:



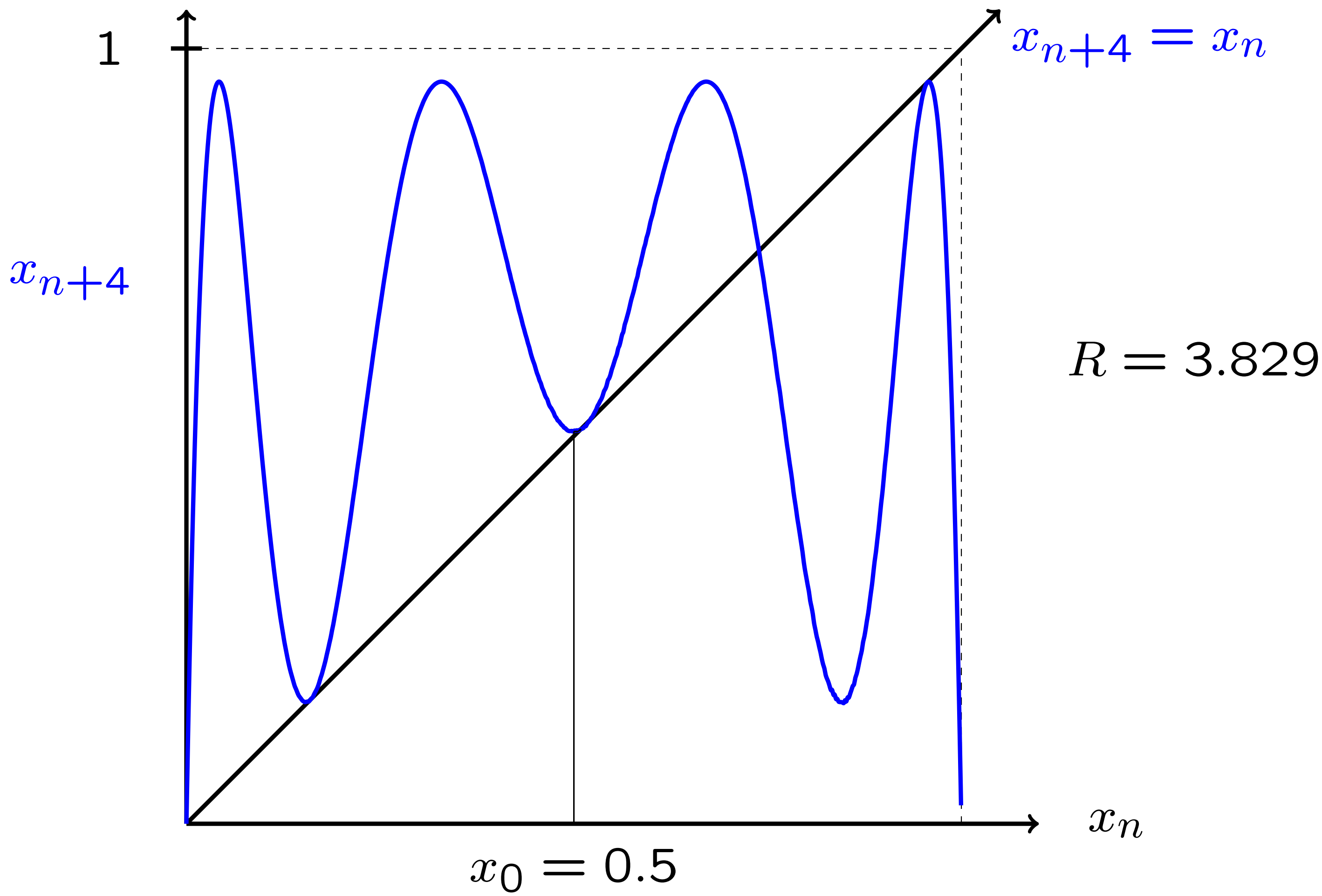
For various values of R:



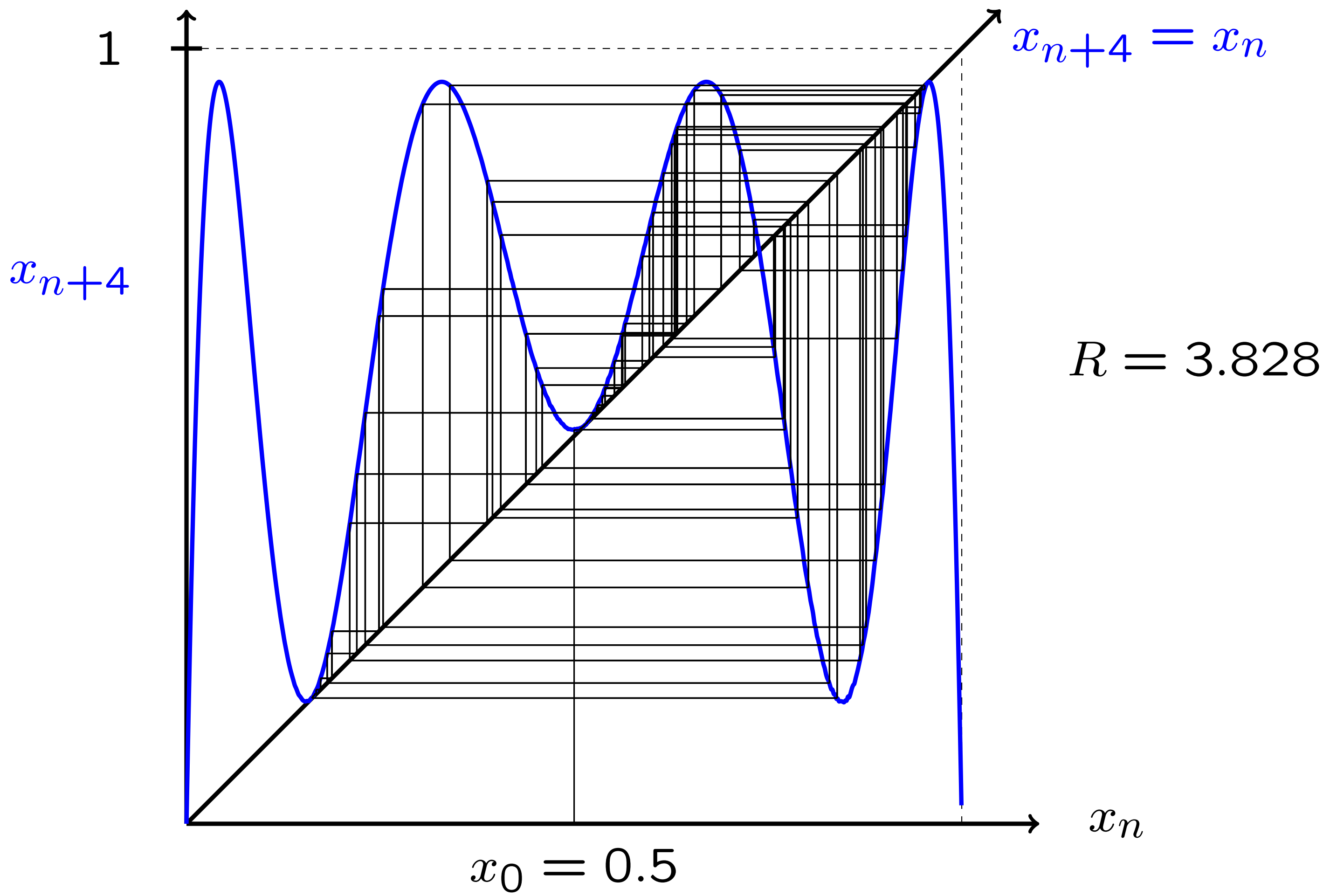
We can do cobweb diagrams also:



With changing starting values:



And changing values of R:



Appendix 1 ←

Just to check (and practice our derivatives :-) with the damped harmonic oscillator example . . .

We had:

$$x(t) = e^{-\frac{b}{2}t} * 2 * \cos\left(\frac{\sqrt{4-b^2}}{2}t\right)$$

from which,

$$\begin{aligned} v(t) = \dot{x}(t) &= -\frac{b}{2}e^{-\frac{b}{2}t} * 2\cos\left(\frac{\sqrt{4-b^2}}{2}t\right) \\ &\quad - e^{-\frac{b}{2}t} * \sqrt{4-b^2}\sin\left(\frac{\sqrt{4-b^2}}{2}t\right) \\ &= -e^{-\frac{b}{2}t}\left(b * \cos\left(\frac{\sqrt{4-b^2}}{2}t\right) + \sqrt{4-b^2} * \sin\left(\frac{\sqrt{4-b^2}}{2}t\right)\right) \end{aligned}$$

Combining the terms:

$$
\begin{aligned}
&-x(t) - bv(t) \\
&= -e^{-\frac{b}{2}t} * 2\cos\left(\frac{\sqrt{4-b^2}}{2}t\right) \\
&\quad + be^{-\frac{b}{2}t}\left(b * \cos\left(\frac{\sqrt{4-b^2}}{2}t\right) + \sqrt{4-b^2}\sin\left(\frac{\sqrt{4-b^2}}{2}t\right)\right) \\
&= e^{-\frac{b}{2}t}\left((b^2-2)\cos\left(\frac{\sqrt{4-b^2}}{2}t\right) + b\sqrt{4-b^2}\sin\left(\frac{\sqrt{4-b^2}}{2}t\right)\right)
\end{aligned}
$$

And, sure enough, when we calculate $\dot{v}(t)$, we get the same thing:

$$\begin{aligned}
\dot{v}(t) = \frac{b}{2}e^{-\frac{b}{2}t}\left(b * \cos\left(\frac{\sqrt{4-b^2}}{2}t\right) + \sqrt{4-b^2}\sin\left(\frac{\sqrt{4-b^2}}{2}t\right)\right) \\
- e^{-\frac{b}{2}t}\left(-b * \frac{\sqrt{4-b^2}}{2}\sin\left(\frac{\sqrt{4-b^2}}{2}t\right) + \frac{4-b^2}{2}\cos\left(\frac{\sqrt{4-b^2}}{2}t\right)\right) \\
= e^{-\frac{b}{2}t}\left(\frac{b^2}{2}\cos\left(\frac{\sqrt{4-b^2}}{2}t\right) + b\frac{\sqrt{4-b^2}}{4}\sin\left(\frac{\sqrt{4-b^2}}{2}t\right)\right) \\
+ e^{-\frac{b}{2}t}\left(b\frac{\sqrt{4-b^2}}{2}\sin\left(\frac{\sqrt{4-b^2}}{2}t\right) - \frac{4-b^2}{2}\cos\left(\frac{\sqrt{4-b^2}}{2}t\right)\right) \\
= e^{-\frac{b}{2}t}\left((b^2-2)\cos\left(\frac{\sqrt{4-b^2}}{2}t\right) + b\sqrt{4-b^2}\sin\left(\frac{\sqrt{4-b^2}}{2}t\right)\right)
\end{aligned}$$

Appendix 2 ←

Something will go here . . .

←

References

[1] Zhan, Mei-Qin,
Course notes, on Systems of First Order Differential Equations
http://www.unf.edu/~ mzhan/chapter4.pdf

To top ←