Symbolic Dynamics (a brief introduction)

csss06

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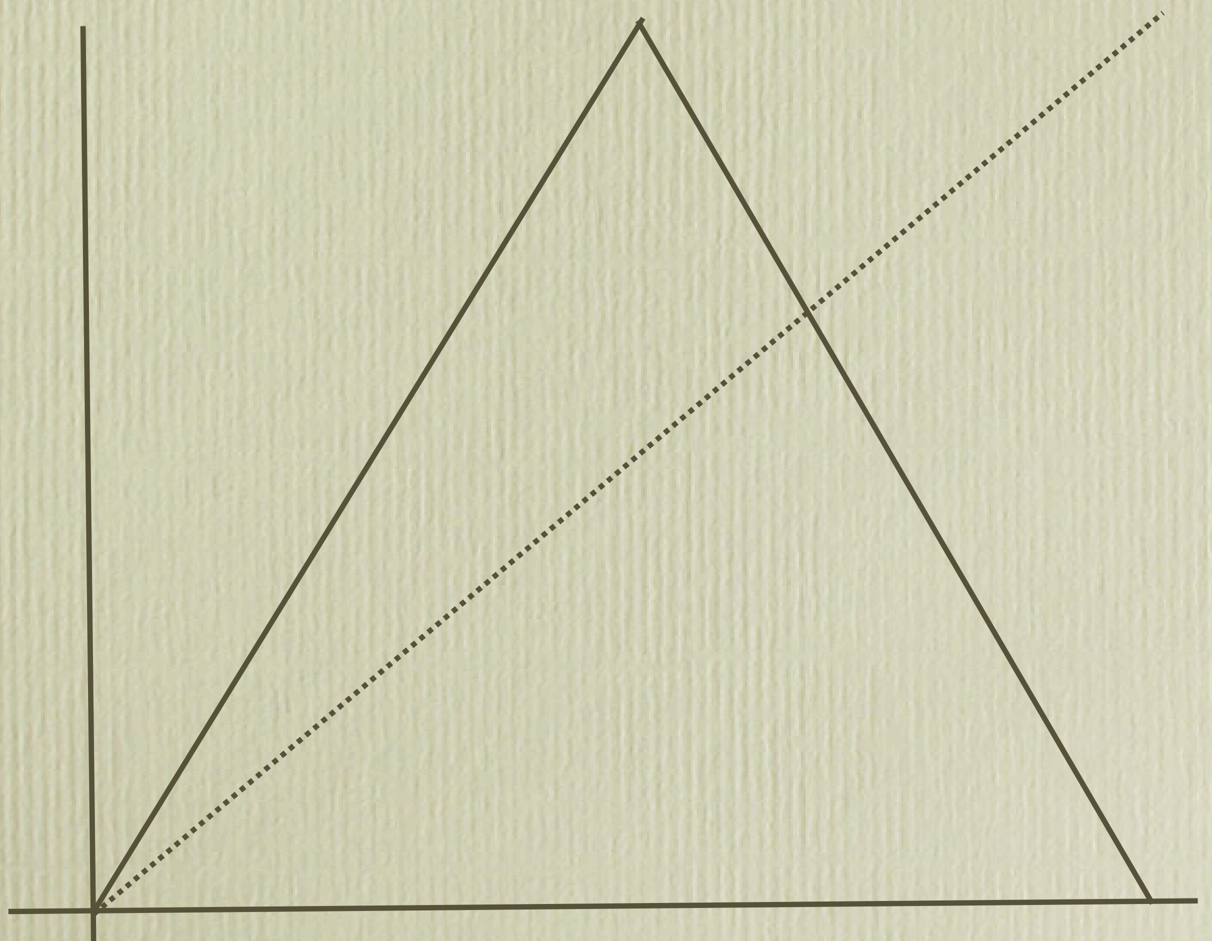
What follows gives one view into an area sometimes called symbolic dynamics

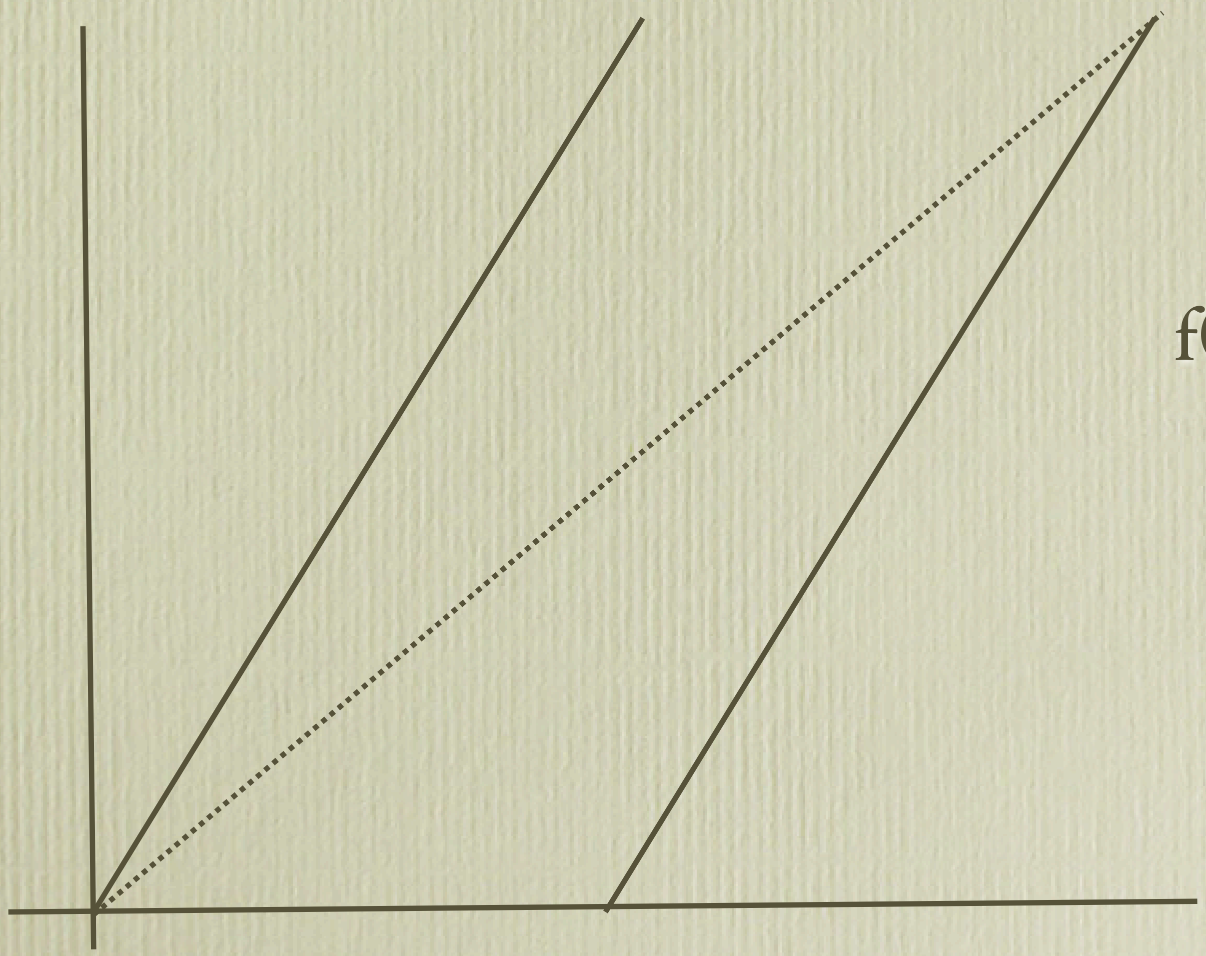
The basic idea is to represent the dynamics of the system using "pure" symbols, as opposed, perhaps, to numeric representations.

Consider a variation of the logistics map:

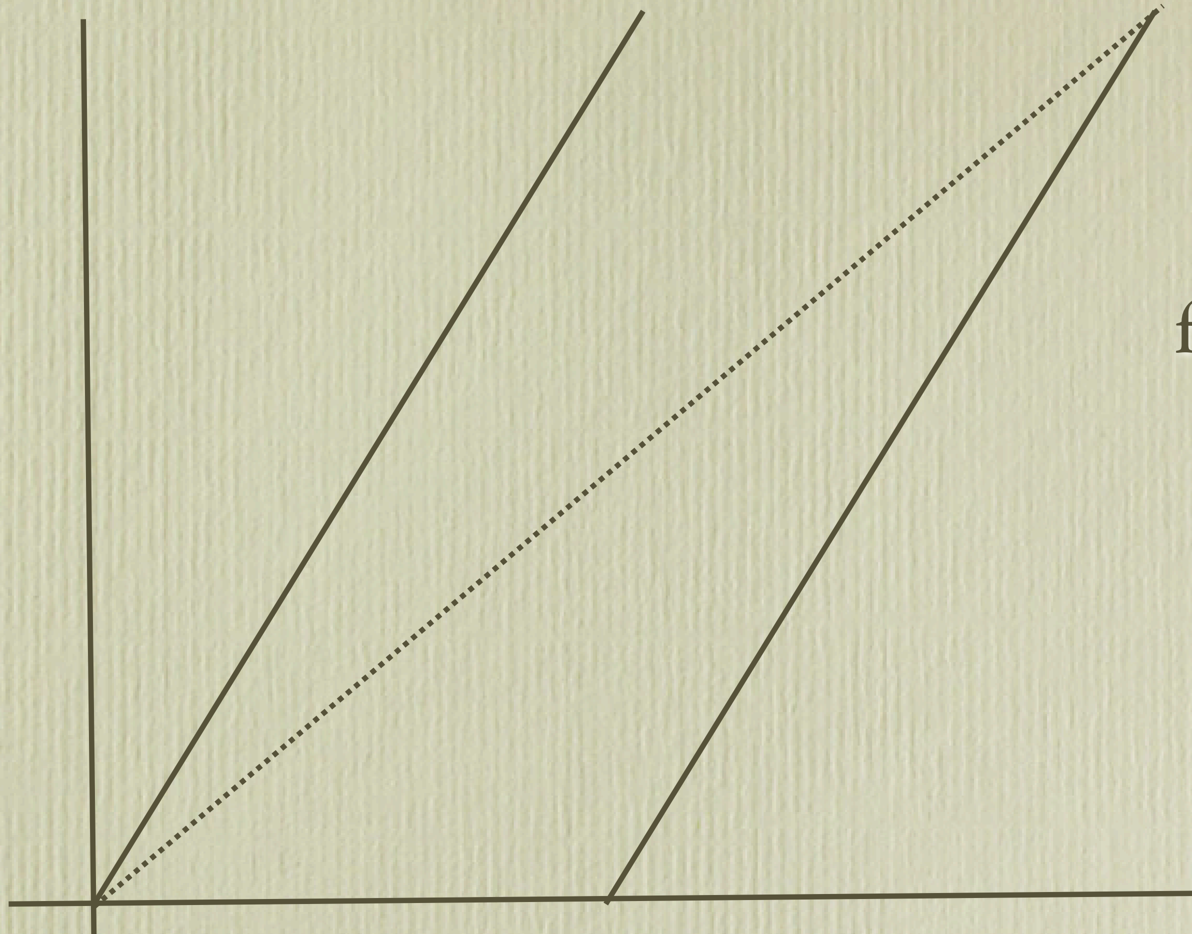


First, we use r = 4, and linearize.

Now, we invert the right half:



$$f(x) = \begin{cases} 2x \text{ if } 0 < x < 1/2 \\ 2x - 1 \text{ if } x >= 1/2 \end{cases}$$



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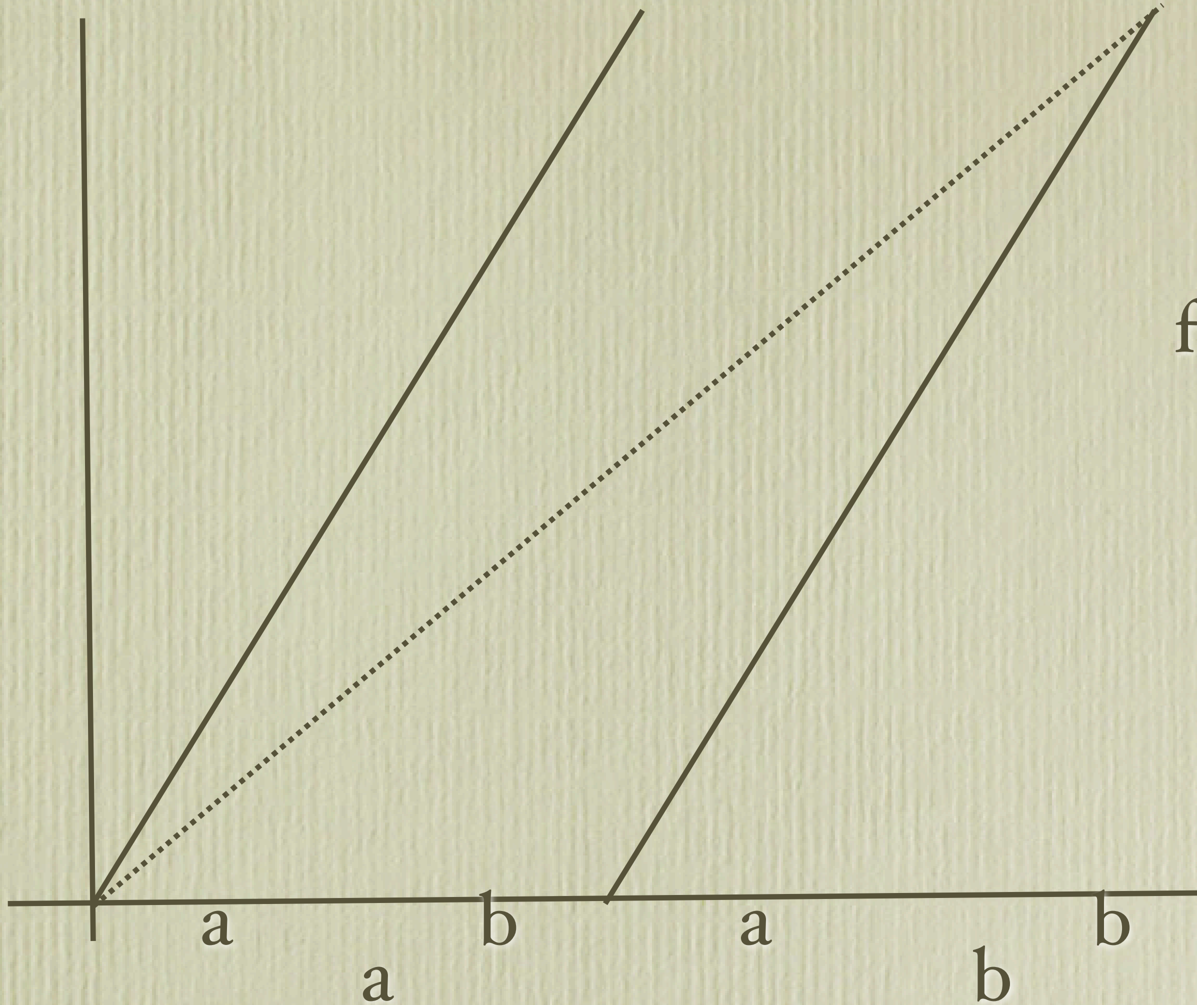
Consider the trajectory of a point (represented in binary notation) x = .001010101001010110101. . . under the map. Successive values of x, f(x), f(f(x)), etc. will be

.01010101001010110101. . ., .1010101001010110101. . .,

.010101001010110101. . ., .10101001010110101. . ., etc.

In other words, we just shift one place to the left.



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If we wanted to, we could represent points in the unit interval as just sequences of symbols (where "a" is for left halves, "b" for right halves: abaababbababbaaabbba . . .
with dynamics being just a shift to the left (discarding the leading symbol). The entire dynamics of the system are captured in this "symbolic" representation.

We can actually learn quite a bit directly from the "symbolic" version

- There are many periodic points (in particular, periodic points are dense). Any repeating pattern is periodic.
- There is "exponential spreading" (sensitive dependence on initial conditions).
- There is topological transitivity (mixing).
- I.e., the map is chaotic on the unit interval

We can also move in the other direction. If we have a system from which we are deriving symbol strings, we may be able to infer information about the dynamics of the system by studying patterns in the strings. For example, this approach is often what is happening with hidden Markov models . . .

This method is an example of a general form of "transformation" of a system from one domain to another, and often back again:

$$\begin{tikzcd} D \arrow[r] \arrow[d] & D \\ S \arrow[r] & S \arrow[u] \end{tikzcd}$$

Theses are sometimes called "commutative diagrams" . . .

There are many possibilities for the horizontal and vertical maps, and often much useful information can be derived by transforming into an alternative domain.

There is a general "theory" of such transformations called Category Theory (also sometimes known as "abstract generalized nonsense" :-)

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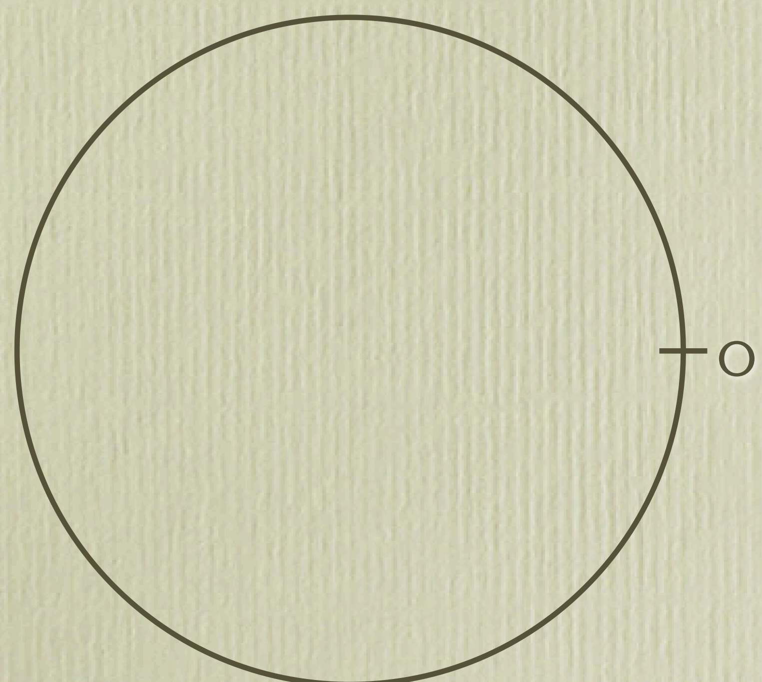
Note also that we can think of the "doubling" map as taking place on the circle, wrapping twice:



$f(\Theta) = 2 * \Theta \pmod{2\pi}$