Voting Paradoxes

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How can we tally votes?

- There are various ways we can handle “voting”
- What ways will accurately reflect the “will of the people”?
Consider the scenario:

- suppose there are four candidates.
- Suppose further (as an example for us to follow) there are 10 voters.
- Suppose also that each voter has a personal linear ranking of the four candidates.
Consider the scenario:

- each voter actually has an opinion on each of the candidates
- no voter has a circular ordering in their preferences
- denote the four candidates by A, B, C, D, and preferences by "\>

Consider the scenario:

- # | ranking
  1 | B > C > A > D
  2 | A > B > C > D
  2 | A > D > B > C
  2 | D > C > B > A
  3 | C > D > B > A
Election results:

- Each voter votes for 1 candidate. The results will be:
  - A : C : D : B
- with vote tallies 4 : 3 : 2 : 1
Consider the scenario:

- # | ranking
  1 | B > C > A > D
  2 | A > B > C > D
  2 | A > D > B > C
  2 | D > C > B > A
  3 | C > D > B > A

(same scenario)
Election results:

- Each voter votes for 2 candidates, results according to total number of votes received:
  - D : C : A : B
  - with vote tallies 7 : 6 : 4 : 3
Consider the scenario:

<table>
<thead>
<tr>
<th>#</th>
<th>ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>B &gt; C &gt; A &gt; D</td>
</tr>
<tr>
<td>2</td>
<td>A &gt; B &gt; C &gt; D</td>
</tr>
<tr>
<td>2</td>
<td>A &gt; D &gt; B &gt; C</td>
</tr>
<tr>
<td>2</td>
<td>D &gt; C &gt; B &gt; A</td>
</tr>
<tr>
<td>3</td>
<td>C &gt; D &gt; B &gt; A</td>
</tr>
</tbody>
</table>
Election results:

- Each voter votes for 3 candidates:
  - B : C : D : A
- with vote tallies 10 : 8 : 7 : 5
Consider the scenario:

- # | ranking
  1 | B > C > A > D
  2 | A > B > C > D
  2 | A > D > B > C
  2 | D > C > B > A
  3 | C > D > B > A

(same scenario)
Election results:

- Each voter expresses their ranking, candidates get 4, 3, 2, or 1 points according to their ranking:
- C : D : B : A
- with vote tallies 27 : 26 : 24 : 23
Consider the scenario:

- # | ranking
  1 | B > C > A > D
  2 | A > B > C > D
  2 | A > D > B > C
  2 | D > C > B > A
  3 | C > D > B > A
Election results:

- Using the four different ways of tallying people's preferences, we got four different results:
  - A : D : B : C
  - B : D : A : C
  - C : D : B : A
  - D : B : A : C
Let's try another one:

- We have three candidates, A, B, and C, for a position.
- The selection committee votes, and finds the ranking A > B > C.
- But, before any offer is made, candidate C withdraws.
Let's try another one:

- Should we just offer the position to A, or should the committee vote again?
Let's try another one:

- Here are the preferences of 13 members on the committee again:

<table>
<thead>
<tr>
<th>#</th>
<th>ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>A &gt; C &gt; B</td>
</tr>
<tr>
<td>4</td>
<td>B &gt; C &gt; A</td>
</tr>
<tr>
<td>3</td>
<td>C &gt; B &gt; A</td>
</tr>
</tbody>
</table>
Let's try another one:

- In the initial vote, the results will be
- \( A > B > C \)
- with vote tallies of 6 : 4 : 3.
Let's try another one:

- After C withdraws, the preferences will be:

<table>
<thead>
<tr>
<th>#</th>
<th>ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>A &gt; B</td>
</tr>
<tr>
<td>4</td>
<td>B &gt; A</td>
</tr>
<tr>
<td>3</td>
<td>B &gt; A</td>
</tr>
</tbody>
</table>
Let's try another one:

- After C withdraws, the preferences will be (or, rather):
  - # | ranking
    6 | A > B
    7 | B > A
Let's try another one:

- After C withdraws, the preferences will be (or, rather):

  - # | ranking
    - 6 | A > B
    - 7 | B > A

- With B the winner . . .
Let's try another one:

- Back to the 13 member committee, with their original preferences:

  - # | ranking
  - 6 | A > C > B
  - 4 | B > C > A
  - 3 | C > B > A
Let's try another one:

- Perhaps we should have used the "points" method in the first place (3 for first, 2 for second, 1 for third). In that case, the results would have been:

- C > A > B

- with points of 29 : 25 : 24
Let's try another one:

- Now, when candidate C drops out, our first choice candidate is no longer available . . .

- Perhaps we should restart the search, and hope for better choices next time . . .
Arrow’s

Impossibility Theorem

- These examples are related to a theorem by economist Kenneth Arrow concerning Social Choice Theory:
  
- No voting system can convert the ranked preferences of individuals into a community-wide ranking (with caveats).