# What shape is a circle?

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# How do we define a circle? ←

We usually define a circle as

$$C = \{\vec{x} \mid ||\vec{x}|| = 1\}$$

(i.e., the set of all vectors $\vec{x}$ of length 1).

Of course, we need to make sense of $||\vec{x}||$, the length of a vector.

Most people start with the definition

$$||\vec{x}|| = \left(\sum_i x_i^2\right)^{\frac{1}{2}},$$

but since we are mathematicians, and don't like our definitions to be too special, we can generalize:

$$||\vec{x}||_p = \left(\sum_i |x_i|^p\right)^{\frac{1}{p}}.$$

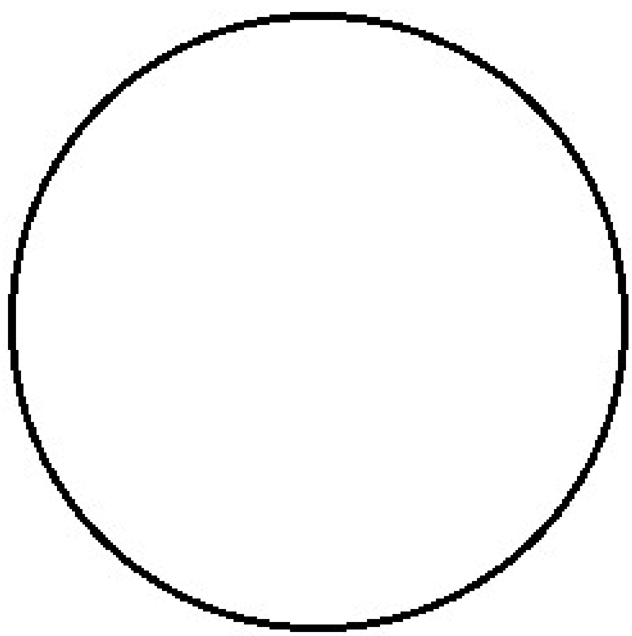
for $p > 0$.

We then have the more general definition of the $p$ circle (in dimension $n$):

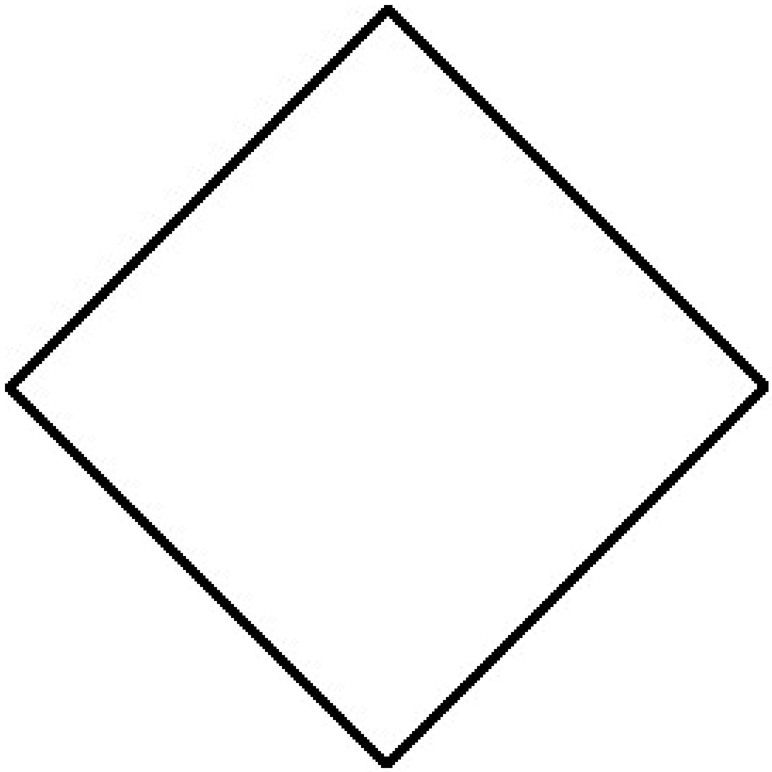
$$C_p^n = \{\vec{x} \mid ||\vec{x}||_p = 1\}.$$

What does $C_p^n$ look like? Let's work in two dimensions, and leave out the dimension label.

$C_2$ is the familiar circle:



$C_1$ is a diamond:



Note that if our vector space is over $\{0, 1\}$, then a vector is just a string of zeros and ones, and $||\vec{x}||_1$ is just the number of ones in the string.

We can convert our length measures into a distance measures:

$$d_p(\vec{x}, \vec{y}) = ||\vec{x} - \vec{y}||_p = \left( \sum_i |x_i - y_i|^p \right)^{\frac{1}{p}}.$$

In particular, if our vector space is over $\{0, 1\}$, then $d_1(\vec{x}, \vec{y})$ is just the Hamming distance between the two vectors (i.e., the number of places in which the two strings differ).

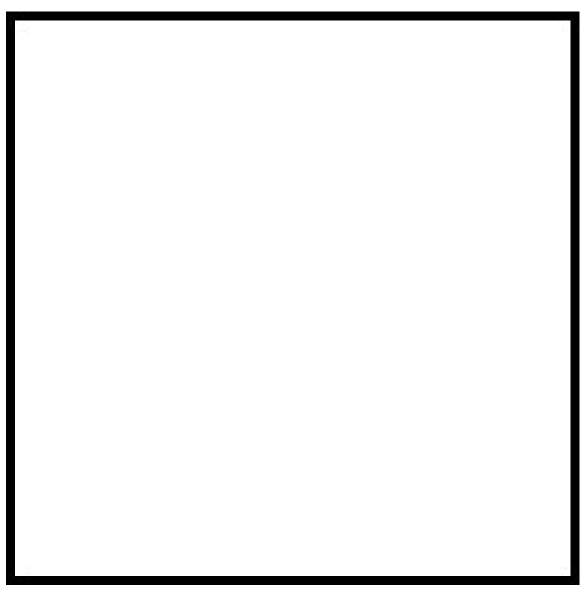
We can also define

$$||\vec{x}||_\infty = \lim_{p \to \infty} (||\vec{x}||_p).$$

Going through the math, we have that

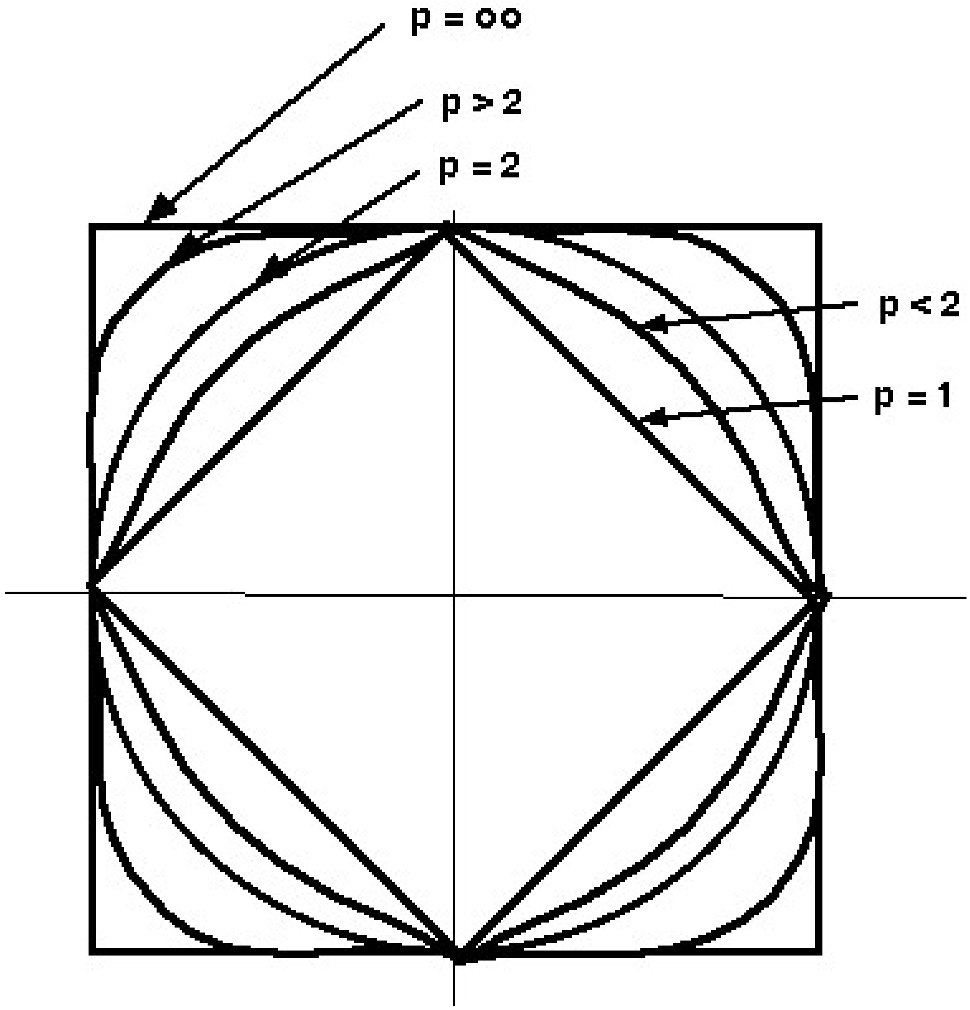
$$||\vec{x}||_\infty = \max_i (|x_i|).$$

We then have that $C_\infty$ is a square:



This means that for a mathematician, a circle is a circle, is a diamond, is a square (which may explain why I always had trouble with those “shape matching” tests . . . :-)

In general, we have the following sort of picture of various circles:



Homework exercises:

What happens for $0 < p < 1$?

What happens if we take the limit as $p$ goes to 0?

Show that in the limit as $p$ goes to 0, the corresponding distance

$$d_0(\vec{x}, \vec{y}) = ||\vec{x} - \vec{y}||_0$$

is a generalized Hamming distance that counts the number of coordinates that are different from each other . . .