The Logistic Flow
(continuous)

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Complex Systems Summer School

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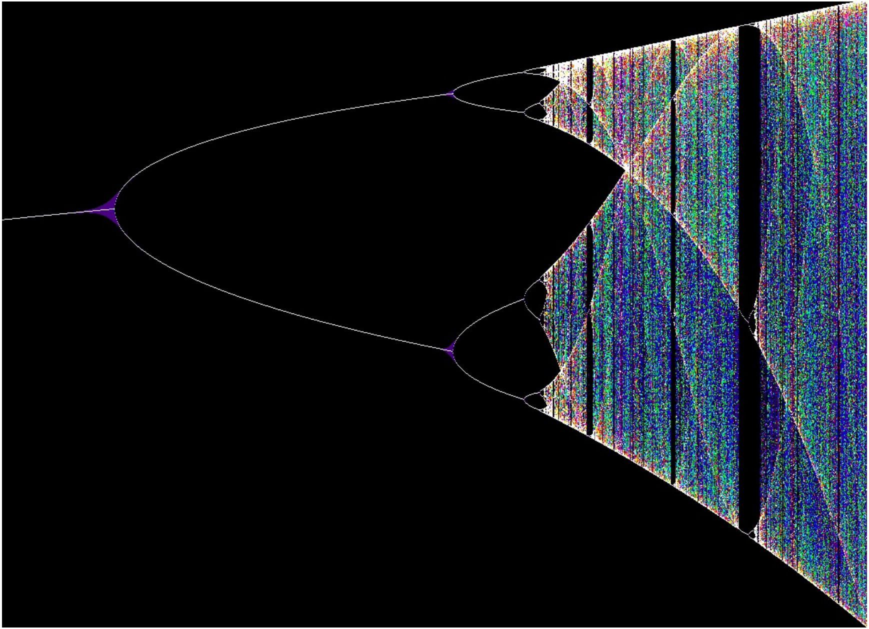
Discrete logistic map

We all know that the discrete logistic map

$$P_{n+1} = rP_n(1 - P_n)$$

exhibits interesting behavior of various sorts for various values of the parameter r, including chaos, etc.

Discrete logistic map – bifurcation diagram



Continuous logistic flow

What kind of behavior can we expect from a continuous version of a logistic flow:

$$\frac{dP}{dt} = rP(1 - P) \quad ?$$

Continuous logistic flow - solving

Note that this is a non-linear ODE, but fortunately we can actually integrate …

$$
\begin{aligned}
\frac{dP}{dt} &= rP(1-P) \\
\frac{dP}{P(1-P)} &= rdt
\end{aligned}
$$

Thus:

$$
\begin{aligned}
\int \frac{dP}{P(1-P)} &= \int rdt \\
\int \frac{dP}{P(1-P)} &= rt + c_1
\end{aligned}
$$

Continuous logistic flow - solving

By partial fractions, we have:

$$\begin{aligned}
\int \frac{dP}{P} + \int \frac{dP}{(1-P)} &= rt + c_1 \\
\log(P) - \log(1-P) &= rt + c_1 \\
\log\left(\frac{P}{1-P}\right) &= rt + c_1 \\
\frac{P}{1-P} &= e^{rt+c_1} \\
\frac{P}{1-P} &= c_2 e^{rt}
\end{aligned}$$

Continuous logistic flow - solving

This gives us:

$$P = (1 - P)c_2 e^{rt}$$

And thus:

$$\begin{aligned} P &= c_2 e^{rt} - P c_2 e^{rt} \\ P + P c_2 e^{rt} &= c_2 e^{rt} \\ P(1 + c_2 e^{rt}) &= c_2 e^{rt} \end{aligned}$$

Continuous logistic flow - solving

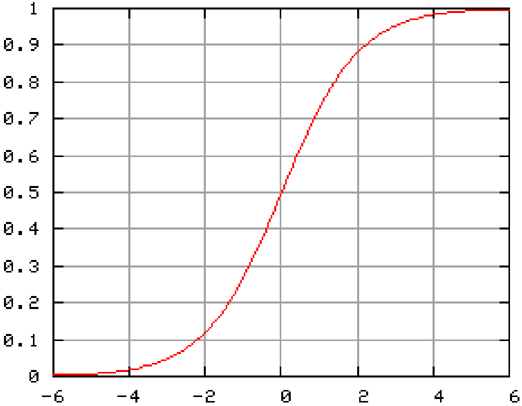
From this we get:

$$P = \frac{c_2 e^{rt}}{1 + c_2 e^{rt}}$$

Finally, dividing top and bottom by $c_2 e^{rt}$ and simplifying, we have:

$$P = \frac{1}{1 + ce^{-rt}}$$

The classic logistic/sigmoid curve



and changes in *c* and *r* make minor changes in the behavior near 0 . . .

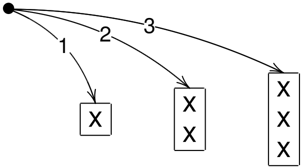
Discrete vs. Continuous

The difference between the behavior of the discrete and continuous logistic functions can give us some idea of the significance of working in the discrete regime . . .

xy-pic test page

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "="] & B \arrow[d] \\
B \arrow[r] & C
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \lambda\omega \arrow[rr, dash] \arrow[dl, dash] \arrow[dd, dash] & & \lambda C \arrow[dl, dash] \arrow[dd, dash] \\
\lambda 2 \arrow[rr, dash] \arrow[dd, dash] & & \lambda P2 \arrow[dd, dash] & \\
& \lambda\underline{\omega} \arrow[rr, dash] \arrow[dl, dash] & & \lambda P\underline{\omega} \arrow[dl, dash] \\
\lambda{\to} \arrow[rr, dash] & & \lambda P &
\end{tikzcd}
$$



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Slides for this talk will be available at:

`http://csustan.csustan.edu/~tom/SFI-CSSS/2009`

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