Stirling's Approximation
(to $n!$)

Stirling's approximation to the factorial is typically written as:

$$n! \;\approx\; \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \tag{1}$$

To find this approximation, we can begin with the observation that:

$$\begin{aligned}
\ln(n!) &= \ln(1 * 2 * 3 * \cdots * n) \\
&= \ln(1) + \ln(2) + \ln(3) + \cdots + \ln(n) \\
&= \sum_{i=1}^{n} \ln(i)
\end{aligned}$$

There are various ways to approximate this sum, some more accurate than others, some easier to compute than others.

One relatively straightforward way to approximate is to use integrals:

$$\begin{aligned} \ln(n!) &= \sum_{i=1}^{n} \ln(i) \\ &\approx \int_1^n \ln(x)dx \\ &= n\ln(n) - n + 1 \end{aligned}$$

Exponentiating each side, we get the first approximation:

$$\begin{aligned} n! &\approx e^{n\ln(n)-n+1} \\ &= \left(e^{\ln(n)}\right)^n e^{-n} e \\ &= n^n e^{-n} e \\ &= e * \left(\frac{n}{e}\right)^n \end{aligned}$$

This is fairly rough, so in practice it makes sense to ignore the factor of e in front, and just use the approximation:

$$n! \approx \left(\frac{n}{e}\right)^n$$

This is good enough for a variety of uses . . .

A more careful derivation of Stirling's approximation (including upper and lower bounds) using infinite series for logarithms instead of integrals follows:

$$\begin{aligned}
\ln n! &= n \ln\ n - \sum_{k=1}^{n-1} k \ln\left(1+\frac{1}{k}\right) \\
&= n \ln n + \sum_{L=1}^{\infty} \left(\sum_{k=1}^{n-1} k(-1)^L \frac{k^{-L}}{L} \right) \\
&= n \ln\ n - (n-1) + \frac{1}{2} \sum_{1}^{n-1} k^{-1} - \frac{1}{3} \sum_{1}^{n-1} k^{-2} + \ldots
\end{aligned}$$

Approximate $\sum k^{-1}$ using

$$\ln\ n = \sum_{1}^{n-1} \ln\left(1+\frac{1}{k}\right) = \sum_{1}^{n-1} k^{-1} - \frac{1}{2} \sum_{1}^{n-1} k^{-2} + \ldots$$

When we group according to powers of k we get:

$$\ln\ n! = n \ln\ n - (n-1) + \frac{1}{2} \ln\ n + \left(\frac{1}{4} - \frac{1}{3}\right) \sum_{1}^{n-1} k^{-2}$$

$$+ \left(-\frac{1}{6} + \frac{1}{4}\right) \sum_{1}^{n-1} k^{-3} + \ldots$$

Let: $\quad S = \left(n + \frac{1}{2}\right) \ln n - (n-1)$

$$M_L = \sum_{k=1}^{\infty} k^{-L}$$

$$M = \sum_{L=2}^{\infty} (-1)^L \left(\frac{1}{L+1} - \frac{1}{2L}\right) M_L$$

$$\begin{aligned}\ln n! &= S - \frac{1}{12}\left(M_2 - \sum_n^{\infty} k^{-2}\right) \\ &\quad - \sum_{L=3}^{\infty} (-1)^L \left(\frac{1}{L+1} - \frac{1}{2L}\right)\left(M_L - \sum_n^{\infty} k^{-L}\right) \\ &= S - M + \frac{1}{12}\sum_n^{\infty} k^{-2} - \frac{1}{12}\sum_n^{\infty} k^{-3} \\ &\quad + \frac{3}{40}\sum_n^{\infty} k^{-4} - \frac{1}{15}\sum_n^{\infty} k^{-5} + \ldots\end{aligned}$$

Since:

$$\frac{1}{12k^2} - \frac{1}{12k^3} + \frac{3}{40k^4} = \frac{1}{12}\left(\frac{1}{k} - \frac{1}{k+1}\right) - \frac{k-9}{120k^4(k+1)},$$

we have

$$S - M + \frac{1}{12n} - \sum_n^{\infty} \frac{k-9}{120k^4(k+1)}$$

$$> \ln\ n! >$$

$$S - M + \frac{1}{12n} - \sum_n^{\infty} \frac{k-9}{120k^4(k+1)} - \sum_n^{\infty} \frac{1}{15k^5}$$

For $n \geq 9$, $\ln n! < S - M + 1/12n$ is immediate.
For a lower bound, we can use $[k \geq 9]$:

$$\frac{k-9}{120k^4(k+1)} + \frac{1}{15k^5} = \frac{k^2-k+8}{120k^5(k+1)}$$

$$< \frac{1}{120k^3(k+1)} < \frac{1}{360}\left(\frac{1}{(k-1)^3} - \frac{1}{k^3}\right)$$

to obtain
$\ln n! > S - M + 1/(12n) - 1/(360(n-1)^3)$.

To determine M, the usual argument involving Wallis' product can be used:

$$\begin{aligned} \lim_{n\to\infty} \frac{4^n(n!)^2}{\sqrt{2n}(2n!)} &= \lim \frac{2\cdot 4\cdot 6\ldots(2n-2)\sqrt{2n}}{3\cdot 5\ldots(2n-1)} \\ &= \sqrt{\frac{\pi}{2}} \\ &= e^{1-\ln 2-M} \end{aligned}$$

So: $e^{-M} = \frac{\sqrt{2\pi}}{e}$