

This derivation of Stirling's approximation (including upper and lower bound) uses infinite series for logarithms instead of integrals.

$$\begin{aligned}
 \ln n! &= n \ln n - \sum_{k=1}^{n-1} k \ln \left(1 + \frac{1}{k}\right) \\
 &= n \ln n + \sum_{L=1}^{\infty} \left(\sum_{k=1}^{n-1} k (-1)^L \frac{k^{-L}}{L} \right) \\
 &= n \ln n - (n-1) + \frac{1}{2} \sum_1^{n-1} k^{-1} - \frac{1}{3} \sum_1^{n-1} k^{-2} + \dots
 \end{aligned}$$

Approximate $\sum k^{-1}$ using

$$\ln n = \sum_1^{n-1} \ln \left(1 + \frac{1}{k}\right) = \sum_1^{n-1} k^{-1} - \frac{1}{2} \sum_1^{n-1} k^{-2} + \dots$$

When we group according to powers of k we get:

$$\ln n! = n \ln n - (n-1) + \frac{1}{2} \ln n + \left(\frac{1}{4} - \frac{1}{3}\right) \sum_1^{n-1} k^{-2} \\ + \left(-\frac{1}{6} + \frac{1}{4}\right) \sum_1^{n-1} k^{-3} + \dots$$

$$\text{Let: } S = \left(n + \frac{1}{2}\right) \ln n - (n - 1)$$

$$M_L = \sum_{k=1}^{\infty} k^{-L}$$

$$M = \sum_{L=2}^{\infty} (-1)^L \left(\frac{1}{L+1} - \frac{1}{2L}\right) M_L$$

$$\begin{aligned}
\ln n! &= S - \frac{1}{12} \left(M_2 - \sum_n^{\infty} k^{-2} \right) \\
&\quad - \sum_{L=3}^{\infty} (-1)^L \left(\frac{1}{L+1} - \frac{1}{2L} \right) \left(M_L - \sum_n^{\infty} k^{-L} \right) \\
&= S - M + \frac{1}{12} \sum_n^{\infty} k^{-2} - \frac{1}{12} \sum_n^{\infty} k^{-3} \\
&\quad + \frac{3}{40} \sum_n^{\infty} k^{-4} - \frac{1}{15} \sum_n^{\infty} k^{-5} + \dots
\end{aligned}$$

Since:

$$\frac{1}{12k^2} - \frac{1}{12k^3} + \frac{3}{40k^4} = \frac{1}{12} \left(\frac{1}{k} - \frac{1}{k+1} \right) - \frac{k-9}{120k^4(k+1)},$$

we have

$$S - M + \frac{1}{12n} - \sum_n^{\infty} \frac{k-9}{120k^4(k+1)}$$

$$> \ln n! >$$

$$S - M + \frac{1}{12n} - \sum_n^{\infty} \frac{k-9}{120k^4(k+1)} - \sum_n^{\infty} \frac{1}{15k^5}$$

For $n \geq 9$, $\ln n! < S - M + 1/12n$ is immediate.
 For a lower bound, we can use [$k \geq 9$]:

$$\begin{aligned} \frac{k-9}{120k^4(k+1)} + \frac{1}{15k^5} &= \frac{k^2 - k + 8}{120k^5(k+1)} \\ &< \frac{1}{120k^3(k+1)} < \frac{1}{360} \left(\frac{1}{(k-1)^3} - \frac{1}{k^3} \right) \end{aligned}$$

to obtain

$$\ln n! > S - M + 1/(12n) - 1/(360(n-1)^3).$$

To determine M , the usual argument involving Wallis' product can be used:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{4^n (n!)^2}{\sqrt{2n} (2n!)} &= \lim \frac{2 \cdot 4 \cdot 6 \dots (2n-2) \sqrt{2n}}{3 \cdot 5 \dots (2n-1)} \\ &= \sqrt{\frac{\pi}{2}} \\ &= e^{1 - \ln 2 - M} \end{aligned}$$

$$\text{So: } e^{-M} = \frac{\sqrt{2\pi}}{e}$$