

# A Simple Random Walk (Drunkard's Walk)

Here we consider a random walk in a one dimension.

We assume that an object (particle) lives on a one-dimensional lattice with lattice constant  $a$  (see Fig. 1). Probability that the object will jump on the right (left) site is  $p$  ( $q$ ) and it is independent of the site number. Also we have normalization condition  $p + q = 1$ .

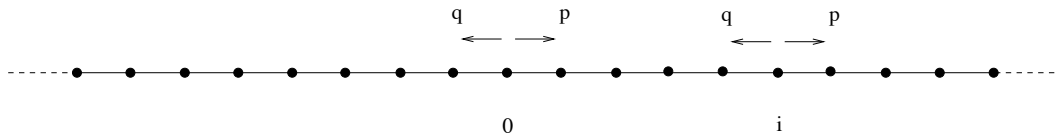


Figure 1: Random walk on a lattice

We are interested in the following probability: The probability of finding the object at position  $r$  in  $N$  steps (jumps) if it starts in the origin.

If an object jumps to the right  $N_r$  times and left  $N_l$  times its coordinate will be  $r = N_r - N_l$  and the probability  $p^{N_r}q^{N_l}$ . It is clear that  $N = N_r + N_l$ .

From above it is clear that there will be many paths leading to the same position and all will have the same probability. Therefore we should count the number of these paths.

The number of such paths is given by the binomial coefficient

$$C_N^{N_r} = \frac{N!}{N_r!N_l!} = \frac{N!}{[(N-r)/2]![(N+r)/2]!}. \quad (1)$$

Therefore the probability will have the following form:

$$P_N(r) = \frac{N!}{[(N-r)/2]![(N+r)/2]!} p^{(N+r)/2} q^{(N-r)/2}. \quad (2)$$

This distribution is called Binomial distribution.

It is easy to show that  $P_N(r)$  is normalized to unity:

$$\sum_{r=-N}^N P_N(r) = \sum_{n=0}^N \frac{N!}{n!(N-n)!} p^n q^{N-n} = (p+q)^N = 1. \quad (3)$$

where  $n = (N+r)/2$ .

## A Random Walk with Reflecting Barriers

An important generalization of the simple random walk can be obtained by including the effects of barriers.

Here we consider an ideal reflecting barrier. By this we mean that if a particle undergoing a one-dimensional random walk arrives at the barrier at position  $m_r$ , it will with certainty be reflected back at the next step, i.e. it will arrive at the position  $m_r - 1$ . We are interested in the following probability: The probability  $P_{N,m_r}(r)$  of finding the object at position  $r$  in  $N$  steps (jumps) if it starts in the origin.

Here we have two different situations: either a particle arrives at  $r$  without any reflections, or it bounces off the barrier one or more times before arriving at  $r$ .

In case of no barriers paths leading to the position  $r < m_r$  cross position  $m_r$  even times (or don't cross at all), while paths leading to the position  $r > m_r$  cross position  $m_r$  odd times. Taking into account that with a barrier we have paths with odd and even reflections, using mirror images of actual paths we can establish correspondence between them and paths with no barrier.

In case of even reflections we construct mirror paths between reflections  $2i - 1$  and  $2i$  ( $i \in \mathbb{Z}$ ). Fig. 2A shows actual and mirror paths in the case of two reflections. The actual paths complemented with mirror images lead to the position  $r$ , therefore contribution from these paths to the  $P_{N,m_r}(r)$  is  $P_N(r)$  ( $r < m_r$ ).

In case of odd reflections we again construct mirror paths between reflections  $2i - 1$  and  $2i$  ( $i \in \mathbb{Z}$ ), but we also construct the mirror path from the last reflection point which leads to the position  $2m_r - r$ . Fig. 2B shows actual and mirror paths in the case of tree reflections. The actual paths complemented with mirror images lead to the position  $2m_r - r$ , therefore contribution from these paths to the  $P_{N,m_r}(r)$  is  $P_N(2m_r - r)$  ( $r < m_r$ ).

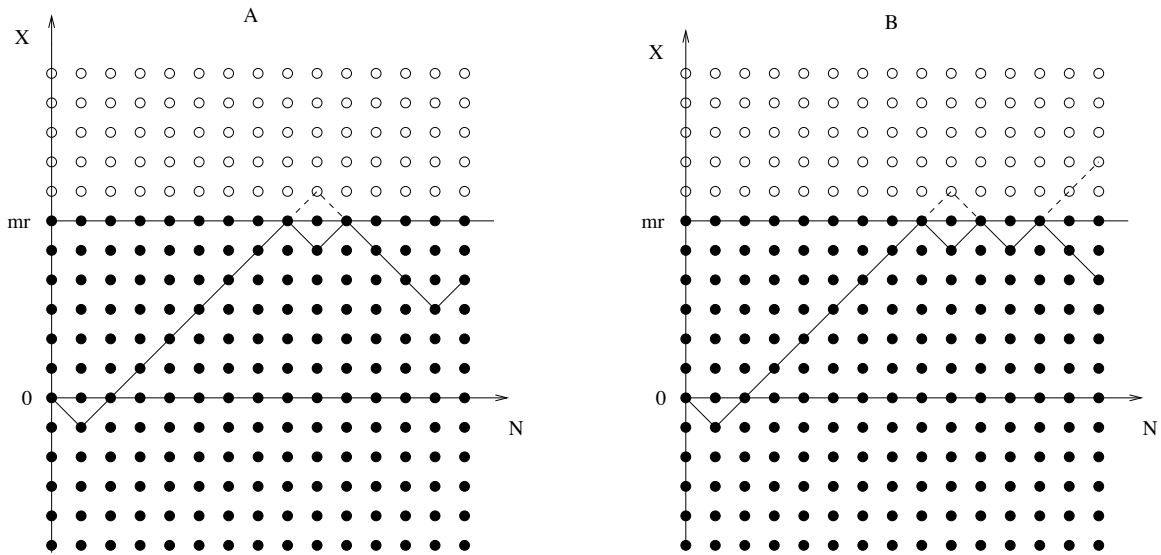


Figure 2: actual (solid line) and mirror (dashed line) paths in the case of two and tree reflections

Finally summing up all the contributions:

$$P_{N,m_r}(r) = P_N(r) + P_N(2m_r - r), \quad (r < m_r). \quad (4)$$

## A Random Walk with Absorbing Barriers

Here we consider an ideal absorbing barrier. By this we mean that if a particle undergoing a one-dimensional random walk arrives at the barrier at position  $m_a$ , it will disappear from the system with certainty. We here also interested in the following probability: The probability  $P_{N,m_a}(r)$  of finding the object at position  $r$  in  $N$  steps (jumps) if it starts in the origin.

Also here we have two different situations: either particle arrives at  $r$  without ever reaching a barrier, or it can arrive at  $m_a$  at an earlier step and be lost from the system before reaching the position  $r$ .

Again here we introduce the mirror paths as we did it in the case of reflecting barrier (see Fig. 2). The present problem can be defined as one in which we take all paths that arrive at  $r$  in  $N$  steps and remove from these all those which at some step encountered the barrier at least once. We know that every path which arrives at the barrier has the unique corresponding path that leads to the position  $2m_a - r$  in the absence of the barrier.

Finally summing up all the contributions:

$$P_{N,m_a}(r) = P_N(r) - P_N(2m_a - r), \quad (r < m_a). \quad (5)$$