IE576 APPLIED STOCHASTIC PROCESSES

Slide Set 3

Thursday, February 26, 1998

1. Markov Chains

A **Markov chain** is a stochastic process

$$\{X_n, n = 0, 1, 2, \ldots\}$$

with finite or countable state space (generally denoted by $\{0, 1, 2, \ldots\}$.

If $X_n = i$, the process is said to be in **state** i.

$$P(X_{n+1} = j \mid X_0 = i_0, X_1 = i_1, \ldots, X_n = i) =$$
$$P(X_{n+1} = j \mid X_n = i)$$

If this is not dependent on n, we say that the Markov Chain is time-homogeneous (stationary) and we simply write

$$P(X_{n+1} = j \mid X_n = i) = P_{ij}$$

These are called the (one-step) **transition probabilities** of the process.

1.1. Note

$$\begin{aligned} P_{ij} &\geq 0 \quad \forall i, j \geq 0 \\ \sum_{j=0}^{\infty} P_{ij} &= 1 \quad \text{for } i = 0, 1, \ldots \end{aligned}$$

Let $\mathbf{P}$ denote the matrix $[P_{ij}]$.

1.2. Example

(Bhat) Professor Null is known for his intemperate moods. If he is in an unpleasant mood during a certain hour, the probability that he will continue to be unpleasant during the following hour is 0.4.

If he is in a pleasant mood during an hour, the probability that he will continue to be pleasant during the following hour is 0.8.

The state space is

$$\begin{aligned} 0 &= \text{unpleasant} \\ 1 &= \text{pleasant} \end{aligned}$$

and the transition probabilities are

$$\begin{aligned} P_{00} = 0.4 \quad P_{01} = 0.6 \\ P_{10} = 0.2 \quad P_{11} = 0.8 \end{aligned}$$

The transition matrix

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix}$$

is then

$$\mathbf{P} = \begin{bmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{bmatrix}$$

Some questions we might like to ask, include:

1. Given that he started the day in an unpleasant mood what is the probability he will be pleasant at the end of the day?
2. Suppose he starts the day in an unpleasant mood. How long does he stay unpleasant, before changing his mood?

3. Suppose he becomes unpleasant at some hour of the day. What is the expected length of time needed for him to become pleasant?

1.3. Example

(Spitzer) Random walk (reflecting barrier)

$$\begin{aligned} P_{0,1} &= 1 \\ P_{i,i+1} &= p > 0 \\ P_{i,i-1} &= q = 1-p \end{aligned}$$

for $i = 1, 2, \ldots$. The transition matrix is of the form

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & & & \\ q & 0 & p & & \cdots \\ & q & 0 & p & \\ & & \vdots & & \end{bmatrix}$$

1.4. Example

Random walk (adsorbing barrier)

$$\begin{aligned} P_{0,0} &= 1 \\ P_{i,i+1} &= p \\ P_{i,i-1} &= q = 1-p \end{aligned}$$

for $i = 1, 2, \ldots$. The transition matrix is of the form

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & & & \\ q & 0 & p & & \cdots \\ & q & 0 & p & \\ & & \vdots & & \end{bmatrix}$$

1.5. Example

Bernoulli process

$$\begin{aligned} P_{i,i+1} &= p \\ P_{i,i} &= q = 1-p \end{aligned}$$

for $i = 0, 1, \ldots$. The transition matrix is of the form

$$\mathbf{P} = \begin{bmatrix} q & p & & & \\ & q & p & & \cdots \\ & & q & p & \\ & & \vdots & & \end{bmatrix}$$

1.6. Example

Gambler's ruin

$$\begin{aligned} P_{i,i+1} &= p \\ P_{i,0} &= q = 1-p \end{aligned}$$

for $i = 0, 1, \ldots$. The transition matrix is of the form

$$\mathbf{P} = \begin{bmatrix} q & p & & & \\ q & 0 & p & & \cdots \\ q & 0 & 0 & p & \\ & & \vdots & & \end{bmatrix}$$

We can define the n-step transition probabilities

$$P(X_{m+n} = j \mid X_m = i) = P_{ij}^{(n)}$$

Note that

$$P_{ij}^{(1)} = P_{ij}$$

1.7. Chapman-Kolmogorov Equations

$$\begin{aligned} P_{ij}^{(n+m)} &= P(X_{m+n} = j \mid X_0 = i) \\ &= \sum_{k=0}^{\infty} P(X_{m+n} = j, X_n = k \mid X_0 = i) \\ &= \sum_{k=0}^{\infty} P(X_{m+n} = j \mid X_n = k, X_0 = i) \\ &\quad \cdot P(X_n = k \mid X_0 = i) \\ &= \sum_{k=0}^{\infty} P_{kj}^{(m)} P_{ik}^{(n)} \end{aligned}$$

And, using matrix notation, we can write

$$\begin{aligned}\mathbf{P}^{(n+m)} &= \mathbf{P}^{(n)}\mathbf{P}^{(m)} \\ \mathbf{P}^{(2)} &= \mathbf{P}^{(1)}\mathbf{P}^{(1)} = \mathbf{P}\mathbf{P} = \mathbf{P}^2\end{aligned}$$

and by induction

$$\mathbf{P}^{(n)} = \mathbf{P}^{(n-1+1)} = \mathbf{P}^{(n-1)}\mathbf{P} = \mathbf{P}^n$$

1.8. Note

We often write

$$P_{ij}^{(n)} = P_{ij}^n$$

1.9. Example

Our professor

$$\begin{aligned}\mathbf{P}^2 &= \begin{bmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{bmatrix}\begin{bmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{bmatrix} \\ &= \begin{bmatrix} 0.28 & 0.72 \\ 0.24 & 0.76 \end{bmatrix}\end{aligned}$$

2. Classification of States

Definition 3.1. *State j is* **accessible** *from state i ($i \to j$) if $P_{ij}^n > 0$ for some $n \geq 0$.*

Two states, i and j **communicate** *($i \leftrightarrow j$) if j is accessible from i, and i is accessible from j.*

2.1. Note

$i \leftrightarrow i$ since

$$P_{ii}^0 = P(X_0 = i \,|\, X_0 = i) = 1$$

Properties

1. $i \leftrightarrow i$ (reflexive)
2. If $i \leftrightarrow j$ then $j \leftrightarrow i$ (symmetric)
3. If $i \leftrightarrow j$ and $j \leftrightarrow k$ then $i \leftrightarrow k$ (transitive)

   Proof: $\exists\, n, m$ such that $P_{ij}^n > 0$ and $P_{jk}^m > 0$. From the Chapman-Kolmogorov equations we have

$$\begin{aligned}P_{ik}^{n+m} &= \sum_{r=0}^{\infty} P_{ir}^n P_{rk}^m \\ &\geq P_{ij}^n P_{jk}^m > 0\end{aligned}$$

■

Given the above, the relation $\leftrightarrow$ is an *equivalence relation.*

Definition 3.2. *Two states that communicate with each other are said to be in the same* **communicating class.**

A Markov chain is **irreducible** *if there is only one communicating class.*

2.2. Example

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

had one communicating class $\{0, 1, 2\}$ and is irreducible.

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

had three communicating classes, $\{0, 1\}$, $\{2\}$ and $\{3\}$.

Definition 3.3. *State i is said to have* **period** *d if $P_{ii}^n = 0$ except when $n = d, 2d, 3d, \ldots$ and d is the greatest integer with this property. We denote the period of state i by $d(i)$. A state with period 1 is said to be* **aperiodic**.

Theorem 3.1. *Periodicity is a class property. That is, if $i \leftrightarrow j$ then $d(i) = d(j)$.*

Proof: $\exists\, m, n$ such that $P_{ij}^m > 0$ and $P_{ji}^n > 0$ Suppose $P_{ii}^s > 0$. Then

$$P_{jj}^{n+s+m} \geq P_{ji}^n P_{ii}^s P_{ij}^m > 0$$

Also, $P_{ii}^s > 0$ implies

$$P_{ii}^{2s} \geq P_{ii}^s P_{ii}^s > 0$$

and, hence

$$P_{jj}^{n+2s+m} > 0$$

Therefore, $d(j)$ divides $n+s+m$ and $n+2s+m$. So, $d(j)$ divides

$$n + 2s + m - (n + s + m) = s$$

whenever $P_{ii}^s > 0$. Therefore, $d(j)$ divides $d(i)$.

Similarly, it can be shown that $d(i)$ divides $d(j)$. Hence $d(i) = d(j)$. ■

2.3. Recurrence

Let

$$\begin{aligned} f_{ii}^{(n)} &\equiv P(\text{first return to state } i \text{ is at time } n) \\ &= P(X_1 \neq i, \ldots, X_{n-1} \neq i, X_n = i \mid X_0 = i) \end{aligned}$$

2.4. Note

For $n \geq 1$

$$P_{ii}^n = \sum_{k=1}^{n} f_{ii}^{(k)} P_{ii}^{n-k}$$

The system "restarts" by going back to i.

2.5. Note

$$P(\text{ever returning to } i \mid X_0 = i) = \sum_{n=1}^{\infty} f_{ii}^{(n)} \equiv f_{ii}^{\star}$$

2.6. Example

Gambler's ruin. Consider random walk on the integers $\{0, 1, \ldots, N\}$ where $\{0, N\}$ are adsorbing states. That is for $p + q = 1$

$$\begin{aligned} P_{i,i+1} &= p \quad \text{for } 0 < i < N \\ P_{i,i-1} &= q \quad \text{for } 0 < i < N \\ P_{0,0} &= 1 \\ P_{N,N} &= 1 \end{aligned}$$

We get

$$\begin{aligned} u_0 &\equiv f_{0N}^{\star} = 0 \\ u_i &\equiv f_{iN}^{\star} = p f_{i+1,N}^{\star} + q f_{i-1,N}^{\star} \qquad i = 1, \ldots, N-1 \\ u_N &\equiv f_{NN}^{\star} = 1 \end{aligned}$$

This produces

$$u_{i+1} - u_i = \frac{q}{p}(u_i - u_{i-1}) \qquad i = 1, \ldots, N-1$$

Therefore

$$\begin{aligned} u_2 - u_1 &= \frac{q}{p} u_1 \\ u_3 - u_2 &= \frac{q}{p}(u_2 - u_1) = \left(\frac{q}{p}\right)^2 u_1 \\ &\vdots \\ 1 - u_{N-1} &= \left(\frac{q}{p}\right)^{N-1} u_1 \end{aligned}$$

Hence

$$u_i - u_1 = u_1 \left[\left(\frac{q}{p}\right) + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^{i-1} \right]$$

for $i > 1$. In other words,

$$u_i = \begin{cases} \dfrac{1 - (q/p)^i}{1 - q/p} u_1 & \text{if } q/p \neq 1 \\ i u_1 & \text{if } q/p = 1 \end{cases}$$

Also, we have

$$u_N = 1 = \begin{cases} \dfrac{1-(q/p)^N}{1-q/p} u_1 & \text{if } q/p \neq 1 \\ N u_1 & \text{if } q/p = 1 \end{cases}$$

hence

$$f^\star_{iN} = u_i = \begin{cases} \dfrac{1-(q/p)^i}{1-(q/p)^N} & \text{if } p \neq 1/2 \\ i/N & \text{if } p = 1/2 \end{cases}$$

If the adversary has an infinite fortune, we can let $N \to \infty$ to get

$$f^\star_{i\infty} = \begin{cases} 1-(q/p)^i & \text{if } p > 1/2 \\ 0 & \text{if } p \leq 1/2 \end{cases}$$

Definition 3.4. *The* **mean recurrence time** *is given by*

$$\mu_i \equiv \sum_{n=1}^{\infty} n f_{ii}^{(n)}$$

Definition 3.5. *State i is said to be* **recurrent** *if $f^\star_{ii} = 1$ and* **transient** *if $f^\star_{ii} < 1$.*

Definition 3.6. *A recurrent state i is said to be* **null recurrent** *if $\mu_i = \infty$ and* **positive recurrent** *if $\mu_i < \infty$.*

Recall the following

Definition 3.7. *Let X be a discrete random variable with support on the nonnegative integers $\{0, 1, 2, \ldots\}$. The* **probability generating function (pgf)** *for X is given by*

$$g(z) \equiv E(z^X) = \sum_{n=0}^{\infty} z^n P(X = n)$$

for $|z| < 1$.

Define

$$\begin{aligned} P_{ii}(z) &\equiv \sum_{n=0}^{\infty} P_{ii}^n z^n & |z| < 1 \\ F_{ii}(z) &\equiv \sum_{n=0}^{\infty} f_{ii}^{(n)} z^n & |z| < 1 \end{aligned}$$

Note that

$$\begin{aligned} i \text{ is recurrent} \quad &\Leftrightarrow \quad F_{ii}(1) = 1 \\ i \text{ is transient} \quad &\Leftrightarrow \quad F_{ii}(1) < 1 \end{aligned}$$

Recall

$$\begin{aligned} P_{ii}^n &= \sum_{k=1}^{n} f_{ii}^{(k)} P_{ii}^{n-k} \qquad n \geq 1 \\ f_{ii}^{(0)} &\equiv 0 \end{aligned}$$

Note however, that for the case $n = 0$ we get

$$\begin{aligned} P_{ii}^0 &= \sum_{k=0}^{0} f_{ii}^{(k)} P_{ii}^{0-k} \\ 1 &= 0 \end{aligned}$$

which needs to be patched. Therefore, we use

$$P_{ii}^n = \delta_n + \sum_{k=0}^{n} f_{ii}^{(k)} P_{ii}^{n-k}$$

for $n \geq 0$, where

$$\delta_n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

From this, we get the probability generating function

$$P_{ii}(z) = 1 + F_{ii}(z) P_{ii}(z)$$

The above is true because if

$$\begin{aligned} A(z) &= \sum_{k=0}^{\infty} a_k z^k \\ B(z) &= \sum_{\ell=0}^{\infty} b_\ell z^\ell \\ C(z) = A(z)B(z) &= \sum_{r=0}^{\infty} c_r z^r \end{aligned}$$

for $|z| < 1$, then

$$c_r = a_0 b_r + a_1 b_{r-1} + \cdots + a_r b_0$$

Since we have

$$P_{ii}(z) = 1 + F_{ii}(z)P_{ii}(z)$$

we get

$$\begin{aligned} F_{ii}(z) &= \frac{P_{ii}(z) - 1}{P_{ii}(z)} \\ F_{ii}(z) &= 1 - \frac{1}{P_{ii}(z)} \end{aligned}$$

Taking the limit on z from below

$$\lim_{z\uparrow 1} F_{ii}(z) \equiv F_{ii}(1) = 1 - \frac{1}{P_{ii}(1)}$$

Hence,

$$F_{ii}(1) \begin{cases} < 1 & \text{if } P_{ii} < \infty \ (i \text{ transient}) \\ = 1 & \text{if } P_{ii} = \infty \ (i \text{ recurrent}) \end{cases}$$

Therefore

$$\begin{aligned} i \text{ transient} &\Leftrightarrow \sum_{n=0}^{\infty} P_{ii}^n(1) < \infty \\ i \text{ recurrent} &\Leftrightarrow \sum_{n=0}^{\infty} P_{ii}^n(1) = \infty \end{aligned}$$

2.7. Note

One must show (see text)

$$\begin{aligned} \sum_{n=0}^{\infty} f_{ii}^{(n)} &= \lim_{z\uparrow 1} F_{ii}(z) \\ &= \lim_{z\uparrow 1} \sum_{n=0}^{\infty} f_{ii}^{(n)} z^n \\ &= \sum_{n=0}^{\infty} \lim_{z\uparrow 1} f_{ii}^{(n)} z^n \end{aligned}$$

Theorem 3.2. *Transience and recurrence are both class properties*

Proof: We have

$$P_{jj}^{m+n+v} \geq P_{ji}^m P_{ii}^n P_{ij}^v$$

If $i \leftrightarrow j$ then we can choose

$$\begin{array}{lll} m & \text{such that} & P_{ji}^m > 0 \\ v & \text{such that} & P_{ij}^v > 0 \end{array}$$

Summing both sides

$$\begin{aligned} \sum_{n=0}^{\infty} P_{jj}^n &\geq \sum_{n=0}^{\infty} P_{jj}^{m+n+v} \\ &\geq P_{ji}^m \left(\sum_{n=0}^{\infty} P_{ii}^n \right) P_{ij}^v \end{aligned}$$

Now suppose that state i is recurrent, then

$$\sum_{n=0}^{\infty} P_{ii}^n = \infty \Rightarrow \sum_{n=0}^{\infty} P_{jj}^n = \infty \Rightarrow j \text{ recurrent}$$

Similarly, if state j is transient, then

$$\sum_{n=0}^{\infty} P_{jj}^n < \infty \Rightarrow \sum_{n=0}^{\infty} P_{ii}^n < \infty \Rightarrow i \text{ transient}$$

■

2.8. Example

Random walk (text page 67)

- **One dimensional** random walk on the integers. At each transition, the process moves right one unit with probability p and left one unit with probability $q = 1 - p$. State 0 (and all states) is recurrent, if and only if $p = \frac{1}{2}$.
- **Two dimensional** random walk with probabilities of right, left, up and down each equal to $\frac{1}{4}$ (symmetric random walk). State 0 (and all states) is recurrent.
- **Three dimensional** (symmetric) random walk with probability $\frac{1}{6}$ of going in each direction. State 0 (and all states) is transient.