How would one define:

$$\frac{d^a(f(t))}{dt^a}$$

For an arbitrary real (or even complex) number a?

Note: we probably at least want our new definition to agree with the old one wherever it can, and to have:

$$\frac{d^a}{dt^a}\left(\frac{d^b(f(t))}{dt^b}\right) = \frac{d^{a+b}(f(t))}{dt^{a+b}}.$$

Consider this:

If

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i s t} dt$$

is the Fourier transform of the function $f(t)$, then we have

$$f(t) = \int_{-\infty}^{\infty} F(s) e^{2\pi i s t} ds.$$

(The inverse transform of the transform of a function is the original function back again.)

Hence, taking derivatives with respect to t of each side, we have

$$f'(t) = \int_{-\infty}^{\infty} 2\pi i s F(s) e^{2\pi i s t} ds.$$

Taking another derivative, we have

$$f''(t) = \int_{-\infty}^{\infty} (2\pi i s)^2 F(s) e^{2\pi i s t} ds,$$

and, in general, for the n^{th} derivative,

$$f^{(n)}(t) = \int_{-\infty}^{\infty} (2\pi i s)^n F(s) e^{2\pi i s t} ds.$$

Homework: Check this.

So, how about if we define:

$$\frac{d^a f(t)}{dt^a} = \int_{-\infty}^{\infty} (2\pi i s)^a F(s) e^{2\pi i s t} ds.$$

Note that for any positive whole number n, this definition agrees with the traditional definition of the derivative.

Quick sanity checks: Do we have

$$\frac{d^0 f(t)}{dt^0} = f(t)?$$

Do we have

$$\frac{d^a}{dt^a}\left(\frac{d^b(f(t))}{dt^b}\right) = \frac{d^{a+b}(f(t))}{dt^{a+b}}?$$

What about negative a? We have that if $G(s)$ is the Fourier transform of $f'(t)$, then

$$G(s) = 2\pi i s F(s)$$

(homework: check this ...)

and hence:

$$\begin{aligned}
\frac{d^{-1}f'(t)}{dt^{-1}} &= \int_{-\infty}^{\infty}(2\pi i s)^{-1}G(s)e^{2\pi i s t}ds \\
&= \int_{-\infty}^{\infty}(2\pi i s)^{-1}(2\pi i s)F(s)e^{2\pi i s t}ds \\
&= \int_{-\infty}^{\infty}F(s)e^{2\pi i s t}ds \\
&= f(t).
\end{aligned}$$

In other words, for nice functions (there may be details to worry about), we have roughly:

$$\frac{d^{-1}f}{dt^{-1}} = \int f$$

(i.e., $\frac{d^{-1}}{dt^{-1}}$ acts like the integral ...). (Note: this means we also got fractional integrals for free, right? :-)

Homework question: How could we interpret

$$\frac{d^i f(t)}{dt^i} \quad ?$$

More homework:

1. Is this operation linear? I.e., do we have:

$$\frac{d^a(c * f(t))}{dt^a} = c * \frac{d^a f(t)}{dt^a}$$

and

$$\frac{d^a(f(t) + g(t))}{dt^a} = \frac{d^a f(t)}{dt^a} + \frac{d^a g(t)}{dt^a}?$$

2. How does this operation work with products? I.e., what can we say about

$$\frac{d^a(f(t) * g(t))}{dt^a}?$$

Note: If your response to all this is

“Cool!!!!!”

then you have the makings of a mathematician!

(As you might expect, I won’t go into what it means if your response is “But what is it good for?” – and I’m confident by now that nobody’s response is “hunh?” ...)

:-)