Voting Paradoxes

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Winter, 2009

How can we tally votes?

- There are various ways we can handle "voting"
- What ways will accurately reflect the "will of the people"?

Consider the scenario:

- suppose there are four candidates.
- Suppose further (as an example for us to follow) there are 10 voters.
- Suppose also that each voter has a personal linear ranking of the four candidates

Consider the scenario:

- each voter actually has an opinion on each of the candidates
- no voter has a circular ordering in their preferences
- denote the four candidates by A, B, C, D, and preferences by ">"

Consider the scenario:

- # | ranking

  1 | B > C > A > D

  2 | A > B > C > D

  2 | A > D > B > C

  2 | D > C > B > A

  3 | C > D > B > A

Election results:

- Each voter votes for 1 candidate. The results will be:
- A : C : D : B
- with vote tallies 4 : 3 : 2 : 1

Consider the scenario:

(same scenario)

#	ranking
1	B > C > A > D
2	A > B > C > D
2	A > D > B > C
2	D > C > B > A
3	C > D > B > A

Election results:

- Each voter votes for 2 candidates, results according to total number of votes received:
- D : C : A : B
- with vote tallies 7 : 6 : 4 : 3

Consider the scenario:

(same scenario)

#	ranking
1	B > C > A > D
2	A > B > C > D
2	A > D > B > C
2	D > C > B > A
3	C > D > B > A

Election results:

- Each voter votes for 3 candidates:
- B : C : D : A
- with vote tallies 10 : 8 : 7 : 5

Consider the scenario:

(same scenario)

#	ranking
1	B > C > A > D
2	A > B > C > D
2	A > D > B > C
2	D > C > B > A
3	C > D > B > A

Election results:

- Each voter expresses their ranking, candidates get 4, 3, 2, or 1 points according to their ranking:
- C : D : B : A
- with vote tallies 27 : 26 : 24 : 23

Consider the scenario:

(same scenario)

#	ranking
1	B > C > A > D
2	A > B > C > D
2	A > D > B > C
2	D > C > B > A
3	C > D > B > A

Election results:

- Using the four different ways of tallying people's preferences, we got four different results:
- A : D : B : C
- B : D : A : C
- C : D : B : A
- D : B : A : C

Let's try another one:

- We have three candidates, A, B, and C, for a position.
- The selection committee votes, and finds the ranking A > B > C.
- But, before any offer is made, candidate C withdraws.

Let's try another one:

- Should we just offer the position to A, or should the committee vote again?

Let's try another one:

- Here are the preferences of 13 members on the committee again:
-

#	ranking
6	A > C > B
4	B > C > A
3	C > B > A

Let's try another one:

- In the initial vote, the results will be
- A > B > C
- with vote tallies of 6 : 4 : 3.

Let's try another one:

- After C withdraws, the preferences will be:
#	ranking
6	A > B
4	B > A
3	B > A

Let's try another one:

- After C withdraws, the preferences will be (or, rather):
- \# | ranking
  6 | A > B
  7 | B > A

Let's try another one:

- After C withdraws, the preferences will be (or, rather):
- \# | ranking
  6 | A > B
  7 | B > A
- With B the winner . . .

Let's try another one:

- Back to the 13 member committee, with their original preferences:
- \# | ranking

#	ranking
6	A > C > B
4	B > C > A
3	C > B > A

Let's try another one:

- Perhaps we should have used the "points" method in the first place (3 for first, 2 for second, 1 for third). In that case, the results would have been:
- C > A > B
- with points of 29 : 25 : 24

Let's try another one:

- Now, when candidate C drops out, our first choice candidate is no longer available . . .
- Perhaps we should restart the search, and hope for better choices next time . . .

Arrow's Impossibility Theorem

- These examples are related to a theorem by economist Kenneth Arrow concerning Social Choice Theory:
- No voting system can convert the ranked preferences of individuals into a community-wide ranking (with caveats).